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Morrey regularity results for asymptotically convex variational problems with (p,q) growth ☆

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ABSTRACT

We prove some global Morrey regularity results for almost minimizers of functionals of the form

$$u \mapsto \int_\Omega f(x, \nabla u)\,dx.$$

This regularity is valid up to the boundary, provided the boundary data is sufficiently regular. The main assumption on f is that for each x, the function $f(x,\cdot)$ behaves asymptotically like a convex function with (p,q) growth. Some discontinuous behavior in the first argument is allowed. As a main application, we establish analogous regularity results for a broad class of systems of nonhomogeneous partial differential equations.

1. Introduction

In this paper, we will prove that almost minimizers for functionals of the form

$$u \mapsto \int_\Omega f(x, \nabla u)\,dx \tag{1}$$

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enjoy Sobolev–Morrey regularity (i.e. both u and ∇u belong to certain Morrey spaces). Our results are global in nature, and are valid up to the boundary of the domain Ω, if the boundary and boundary conditions are smooth enough. The primary assumption on f is that for each x, the function $f(x,\cdot)$ behaves asymptotically like a convex, radial function g with (p,q) growth. We impose no continuity on $f(\cdot,F)$ when $|F|$ is small, and some discontinuity is allowed even when $|F|$ is large. Using this result, we then establish analogous Sobolev–Morrey regularity results for weak solutions to systems of partial differential equations with the form $\operatorname{div}[\mathcal{A}(x,\nabla u)] = h(x,\nabla u)$. In addition to some growth assumptions, we suppose that $F \mapsto \mathcal{A}(x,F)$ behaves asymptotically like $F \mapsto \frac{\partial}{\partial F} g(x,F)$, where g possesses the properties described above.

The integrands for the functionals we consider behave asymptotically like a function with (p,q)-structure (Definition 2.6). By way of example, for fixed $1 < p \leqslant q$, define the function $g : [0,\infty) \to [0,\infty)$ by

$$g(t) := \begin{cases} t^p & \text{if } 0 \leqslant t \leqslant t_0, \\ t^{\frac{p+q}{2} + \frac{p-q}{2} \sin\log\log\log t} & \text{if } t > t_0, \end{cases} \tag{2}$$

where $t_0 > 0$ is chosen so that $\sin\log\log\log t_0 = 1$ (this function was first given as an example in [3]). By a direct computation, one can show that it is possible to choose t_0 large enough so that g is convex and has $(\bar{p},\bar{q})$-structure for some $\bar{p}$ and $\bar{q}$ satisfying $1 < \bar{p} < p \leqslant q < \bar{q}$. It is clear that $g(t)$ oscillates between t^p and t^q, and therefore does not lend itself to the setting of natural growth (i.e. where the upper and lower growth exponents are equal).

Let us say that a function $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^* := \mathbb{R} \cup \{+\infty\}$ is asymptotically convex if there is a function g with (p,q)-structure such that for every $\varepsilon > 0$ and $x \in \Omega$, there is a $\sigma_\varepsilon(x)$ so that

$$\bigl|f(x,F) - g\bigl(|F|\bigr)\bigr| < \varepsilon g\bigl(|F|\bigr),$$

whenever $|F| > \sigma_\varepsilon(x)$. We note here that it is possible that f, though asymptotically convex, is nevertheless not even locally convex at any single point. For example, if we take g as defined in (2) and define $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ by $f(F) := g(|F|) - |F|\chi_{\mathbb{Q}}(|F|)$, it is clear that f is nowhere locally convex, yet it is not difficult to verify that f is asymptotically convex with $\sigma_\varepsilon = \varepsilon^{-1/(p-1)}$.

Our main result is stated in Section 6. It is variational in nature but sufficiently general to provide significant results in the context of systems of partial differential equations. The following theorem is a simplified version of this result.

Theorem 1.1. *Let $n \geqslant 2$, $\Omega \subset \mathbb{R}^n$ be open and bounded, and $0 \leqslant \kappa < n$. Suppose that the function $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^*$ satisfies the following*:

(i) *There are numbers $1 < p \leqslant q$ and a function g with (p,q)-structure such that for every $\varepsilon > 0$, there is a $\sigma_\varepsilon \in L^{q,\kappa}(\Omega)$ so that*

$$\bigl|f(x,F) - g\bigl(|F|\bigr)\bigr| < \varepsilon g\bigl(|F|\bigr)$$

for all $x \in \Omega$ and $F \in \mathbb{R}^{N\times n}$ satisfying $|F| > \sigma_\varepsilon(x)$.

(ii) *There is a constant $L \geqslant 1$ and a function $\alpha \in L^{1,\kappa}(\Omega)$ such that*

$$\bigl|f(x,F)\bigr| \leqslant L|F|^q + \alpha(x)$$

for every $x \in \Omega$ and $F \in \mathbb{R}^{N\times n}$.

Define the functional $K : W^{1,1}(\Omega;\mathbb{R}^N) \to \mathbb{R}^*$ by $K(w) := \int_\Omega f(x,\nabla w)\,\mathrm{d}x$. Then the following hold:

(I) *If $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ is a local minimizer for K, i.e. $K(u) \leqslant K(u+\varphi)$ for every $\varphi \in W^{1,1}(\Omega;\mathbb{R}^N)$ with $\operatorname{supp}(\varphi) \Subset \Omega$, then $u \in W^{1,(p,\kappa)}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$.*

(II) *Suppose further that Ω has a $\mathcal{C}^1$ boundary and that $\bar{u} \in W^{1,(q,\kappa)}(\Omega;\mathbb{R}^N)$. If $u \in W^{1,1}(\Omega;\mathbb{R}^N)$ is a minimizer for K over $\bar{u} + W_0^{1,1}(\Omega;\mathbb{R}^N)$, then $u \in W^{1,(p,\kappa)}(\Omega;\mathbb{R}^N)$.*

The result in Section 6 is actually established for a very general class of almost minimizers. It turns out that weak solutions to certain systems of partial differential equations can be shown to be almost minimizers, in our generalized sense, for associated integral functionals. This leads to our main regularity result for solutions to systems of PDEs. Again we state a simplified version; the full result is provided in Section 7.

Theorem 1.2. *Let $n \geqslant 2$ and $0 \leqslant \kappa < n$. Suppose that $\Omega \subset \mathbb{R}^n$ is open, bounded, and has $\mathcal{C}^1$ boundary. Let $g:[0,\infty) \to [0,\infty)$ be a function with (p,q)-structure, with $1 < p \leqslant q$, and suppose that $\mathcal{A}:\Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ and $h:\Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^N$ satisfy the following:*

(i) *For every $\varepsilon > 0$, there is a $\sigma_\varepsilon \in L^{\frac{p(q-1)}{p-1},\kappa}(\Omega)$ so that*

$$\left|\mathcal{A}(x,F) - \frac{\partial}{\partial F} g\big(|F|\big)\right| < \varepsilon g'\big(|F|\big)$$

for all $x \in \Omega$ and $F \in \mathbb{R}^{N\times n}$ satisfying $|F| > \sigma_\varepsilon(x)$.

(ii) *There is a constant $L \geqslant 1$ and a function $\nu \in L^{\frac{p}{p-1},\kappa}(\Omega)$ such that*

$$\big|\mathcal{A}(x,F)\big| \leqslant L|F|^{q-1} + \nu(x) \quad \text{and} \quad \big|h(x,F)\big| \leqslant L|F|^{p-1} + \nu(x)$$

for every $x \in \Omega$ and $F \in \mathbb{R}^{N\times n}$.

Let $\bar{u} \in W^{1,(q,\kappa)}(\Omega;\mathbb{R}^N)$ be given, and suppose that $u \in W^{1,q}(\Omega;\mathbb{R}^N)$ satisfies $u - \bar{u} \in W_0^{1,q}(\Omega;\mathbb{R}^N)$ and that the equality

$$\int_\Omega \mathcal{A}(x,\nabla u) : \nabla\varphi \, \mathrm{d}x = -\int_\Omega h(x,\nabla u)\cdot\varphi \, \mathrm{d}x$$

holds whenever $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^N)$ is such that $g(|\nabla\varphi|) \in L^1(\Omega)$. Then we find that $u \in W^{1,(p,\kappa)}(\Omega;\mathbb{R}^N)$.

In the case where $\partial\Omega$ is not $\mathcal{C}^1$ or $\bar{u}$ is not assumed to belong to $W^{1,(q,\kappa)}$, a local version of Theorem 1.2 remains valid.

As indicated earlier, our results capture regularity in the setting of the Sobolev–Morrey spaces $W^{1,(p,\kappa)}(\Omega;\mathbb{R}^N)$. These spaces were first introduced in the classical paper by Campanato [2], and have been used in various forms in the context of partial differential equations; see for instance [17] or [19]. We note that if Ω is an open and bounded subset of $\mathbb{R}^n$ without internal cusps, and $1 \leqslant p < \infty$ and $0 \leqslant \kappa \leqslant n$ are such that $p + \kappa > n$, then the Sobolev–Morrey space $W^{1,(p,\kappa)}(\Omega;\mathbb{R}^N)$ is embedded in the space of uniformly Hölder continuous functions $\mathcal{C}^{0,1-\frac{n-\kappa}{p}}(\Omega;\mathbb{R}^N)$ (we refer to [13] for more details). Hence, under certain conditions, our results yield global Hölder continuity and can be viewed as low order regularity results. We point out that even in the natural growth setting, there are relatively few low order regularity results available (see [18] and [10] for a discussion).

To place our work in a broader context, we mention a few recent results in the natural growth setting. In [11], the authors provide Sobolev–Morrey regularity for almost minimizers of functionals of the form $u \mapsto \int_\Omega f(x,u,\nabla u)\,\mathrm{d}x$, where $f(x,u,\cdot)$ is asymptotically convex for each $(x,u) \in \Omega \times \mathbb{R}^N$. Our paper provides an extension of their results to allow the more general (p,q) growth condition. Our results are stated and proved for integrands without explicit dependence on u, but this is only to keep the present paper to a more manageable length. We also mention a result by Kristensen and Taheri. In [14], they show that if u is a weak solution to $\operatorname{div}[\mathcal{A}(\nabla u)] = 0$, with $\mathcal{A}$ continuous and satisfying condition (i) in Theorem 1.2 with $g(|F|) := |F|^p$, then $u \in W^{1,r}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ for every $r < \infty$. For the purpose of comparison, under such an assumption, Theorem 1.2 yields $u \in W^{1,(p,\kappa)}(\Omega; \mathbb{R}^N)$ for all $0 \leqslant \kappa < n$; this Sobolev–Morrey regularity is valid up to the boundary $\partial\Omega$. Moreover, as indicated in Theorem 1.2, we are able to treat nonhomogeneous problems and allow discontinuous variable coefficients. It is worth observing that in the scalar setting, where the system reduces to a single equation, it is possible to obtain similar regularity results under more flexible hypotheses. For an extensive treatment of nonlinear elliptic equations with divergence form that includes such results, we refer to the recent work of Mingione [19].

Returning to variational problems with general growth, we point out that some higher integrability results have been obtained in [4–7,12,15]. In each of these papers, the integrands are not required to asymptotically behave in any particular manner but an assumption is made that p and q are not too far apart. In fact, in [5], the authors give an example that shows that there can be local minimizers $u \in W^{1,p}_{\mathrm{loc}}$ that do not belong to $W^{1,q}_{\mathrm{loc}}$ if q/p is too large. We emphasize that the Morrey regularity we obtain requires only that $1 < p \leqslant q$; the ratio q/p does not affect the type of Sobolev–Morrey space to which u belongs.

Our results can also be used to establish some low order regularity up to the boundary for minimizers that until now had only been shown to be locally regular. Consider for the moment the functional $u \mapsto \int_\Omega g(|\nabla u|)\,\mathrm{d}x$, where g is a function with (p,q)-structure. From the results obtained by P. Marcellini and G. Papi in [16], we conclude that a minimizer u will belong to $W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$. If the boundary $\partial\Omega$ is smooth enough and there is a function $\bar{u} \in W^{1,\infty}(\Omega; \mathbb{R}^N)$ with $u = \bar{u}$ on $\partial\Omega$, then combining our results with the result of Marcellini and Papi, we find that $u \in W^{1,\infty}_{\mathrm{loc}} \cap W^{1,(p,\kappa)}(\Omega; \mathbb{R}^N)$ for each $0 \leqslant \kappa < n$. In particular, using the embedding described above, we have $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^N) \cap \mathcal{C}^{0,\alpha}(\overline{\Omega}; \mathbb{R}^N)$ for every $0 \leqslant \alpha < 1$.

We conjecture that it is possible to use our results to prove global Lipschitz regularity of a minimizer. In [9], M. Foss carries this out in the case $2 < p = q$. His proof is similar in spirit to that of J.-P. Raymond [20], where the gradient of a minimizer is first shown to belong to a certain Morrey space. The Morrey regularity is then used as a stepping-stone to show that the minimizer is in fact Lipschitz.

We note that Definition 2.7 does not allow us to consider $p(x)$-growth (e.g., when $f(x,F) = |F|^{p(x)}$, with $1 < p < p(x) < q$), which has applications to electrorheological fluids and other models from mathematical physics. We refer the reader to [18] and the references therein for some results obtained under these conditions. We expect that our definition can be relaxed to allow this type of growth, but doing so introduces some technical issues, particularly in the proof for Lemma 5.3. Because of length considerations, we have elected to study regularity in this setting separately.

Finally, we wish to comment briefly on the local Lipschitz estimates obtained in Section 4. In [16], P. Marcellini and G. Papi prove local Lipschitz regularity, and consequently $\mathcal{C}^k$ and $\mathcal{C}^\infty$ regularity, for minimizers of functionals of the form $u \mapsto \int_\Omega g(|\nabla u|)\,\mathrm{d}x$, where g is only required to satisfy very mild growth conditions. Under the assumption of (p,q) growth, we apply the method used there to obtain a refinement of their result; this refinement yields an estimate of the form

$$\big\| g\big(|\nabla u|\big)\big\|_{L^\infty(\mathcal{B}_{x_0,\rho})} \leqslant \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{x_0,R}} g\big(|\nabla u|\big)\,\mathrm{d}x, \tag{3}$$

which is crucial for our purposes. Until now, estimates of the form (3) have only been available when g satisfies natural growth conditions; additionally, the proofs for these results have been separated into two cases, namely $1 < p < 2$ and $p \geqslant 2$, and the two cases have been proved in fairly different ways (see [1] and [21]). In contrast, the proof given in Section 4 is essentially unified. Though certain growth conditions are implied by our definition of (p,q)-structure (see Lemma 3.1), our proof uses the structure intrinsic to g itself, as opposed to external growth conditions imposed on g. Therefore it seems that similar results could be shown for functions with more general growth by employing techniques similar to those used here.

2. Definitions and notation

Throughout, $\Omega \subset \mathbb{R}^n$ is open, with $n \geqslant 2$. We will use x, y, and z to denote points in $\mathbb{R}^n$, and F to denote a point in $\mathbb{R}^{N\times n}$. The open ball of radius ρ centered at the point x is represented by $\mathcal{B}_{x,\rho}$. For brevity $\mathcal{B}_\rho$ denotes $\mathcal{B}_{0,\rho}$ and $\mathcal{B}$ denotes $\mathcal{B}_1$. We define $\mathcal{H}^+ := \{(x_1,\dots,x_n) \in \mathbb{R}^n\colon\ x_n > 0\}$, and given a set $\mathcal{U} \subset \mathbb{R}^n$, we use $\mathcal{U}^+$ for $\mathcal{U} \cap \mathcal{H}^+$ and $\overline{\mathcal{U}^+}$ to stand for $\mathcal{U} \cap \overline{\mathcal{H}^+}$. We use C to denote a finite, positive constant that, unless otherwise stated, depends only on n, p, and q. The constants p and q serve as structural parameters for the integrands of the functionals we consider here. For convenience, we will always assume that $C \geqslant 1$. The value of C may change from line to line in our computations.

We now recall the definitions for Morrey and Sobolev–Morrey spaces, and for later convenience, we also introduce a notion of Orlicz–Morrey spaces. In these definitions, $\mathcal{U} \subset \mathbb{R}^n$ is a measurable set.

Definition 2.1. For each $p \in [1,\infty)$ and $0 \leqslant \kappa \leqslant n$, we define the *Morrey space*

$$L^{p,\kappa}(\mathcal{U};\mathbb{R}^N) := \left\{ u \in L^p(\mathcal{U};\mathbb{R}^N)\colon\ \sup_{\substack{y\in\mathcal{U}\\ \rho>0}} \frac{1}{\rho^\kappa} \int_{\mathcal{U}\cap\mathcal{B}_{y,\rho}} |u|^p \,\mathrm{d}x < \infty \right\}.$$

We write $u \in L^{p,\kappa}_{\mathrm{loc}}(\mathcal{U};\mathbb{R}^N)$ if $u \in L^{p,\kappa}(\mathcal{U}';\mathbb{R}^N)$ for every $\mathcal{U}' \Subset \mathcal{U}$.

Definition 2.2. For each $p \in [1,\infty)$ and $0 \leqslant \kappa \leqslant n$, we say that a mapping $u \in W^{1,p}(\mathcal{U};\mathbb{R}^N)$ belongs to the *Sobolev–Morrey space* $W^{1,(p,\kappa)}(\mathcal{U};\mathbb{R}^N)$ if $u \in L^{p,\kappa}(\mathcal{U};\mathbb{R}^N)$ and $\nabla u \in L^{p,\kappa}(\mathcal{U};\mathbb{R}^{N\times n})$. We write $u \in W^{1,(p,\kappa)}_{\mathrm{loc}}(\mathcal{U};\mathbb{R}^N)$ if $u \in W^{1,(p,\kappa)}(\mathcal{U}';\mathbb{R}^N)$ for every $\mathcal{U}' \Subset \mathcal{U}$.

Definition 2.3. Suppose that $g \in \mathcal{C}^1([0,\infty))$ is a strictly increasing convex function that satisfies $g(0) = 0$. With $0 \leqslant \kappa \leqslant n$, we say that $u : \mathcal{U} \to \mathbb{R}^N$ belongs to the *Orlicz–Morrey space* $L^{g,\kappa}(\mathcal{U};\mathbb{R}^N)$, if $g(|u|) \in L^{1,\kappa}(\mathcal{U})$. We write $u \in L^{g,\kappa}_{\mathrm{loc}}(\mathcal{U};\mathbb{R}^N)$, if $g(|u|) \in L^{1,\kappa}_{\mathrm{loc}}(\mathcal{U})$.

Along the same lines as [9,11], we introduce the following generalized notion for almost minimizers.

Definition 2.4. Let $\Omega \subset \mathbb{R}^n$ be open, and suppose $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is given. Define the functional $K : W^{1,1}(\Omega;\mathbb{R}^N) \to \mathbb{R}^*$ by

$$K[w] := \int_\Omega f(x, \nabla w)\,\mathrm{d}x.$$

Let $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^1(\Omega)$ and suppose that $\{\omega_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}^0([0,\infty))$ is a family of nondecreasing functions satisfying $\omega_\varepsilon(0) = 0$ for each $\varepsilon > 0$. We say that $u \in W^{1,1}(\Omega;\mathbb{R}^N)$ is a $(K,\{\omega_\varepsilon\},\{\nu_\varepsilon\})$-*minimizer at* y, if $K(u) < \infty$ and for every $\varepsilon > 0$ and $0 < \rho < \operatorname{diam}(\Omega)$, we find that

$$\begin{aligned} K(u) \leqslant K(u+\varphi) &+ \big(\omega_\varepsilon(\rho) + \varepsilon\big) \int_{\Omega\cap\mathcal{B}_{y,\rho}} \big\{ \big|f(x,\nabla u)\big| + \big|f(x,\nabla u + \nabla\varphi)\big| \big\}\,\mathrm{d}x \\ &+ \int_{\Omega\cap\mathcal{B}_{y,\rho}} \big\{ \big|\nu_\varepsilon(x)\big| + \big|\nu_\varepsilon(y)\big| \big\}\,\mathrm{d}x, \end{aligned} \tag{4}$$

whenever $\varphi \in W^{1,1}_0(\Omega \cap \mathcal{B}_{y,\rho};\mathbb{R}^N)$. If u is a $(K,\{\omega_\varepsilon\},\{\nu_\varepsilon\})$-minimizer at every $y \in \Omega$, then we call u a $(K,\{\omega_\varepsilon\},\{\nu_\varepsilon\})$-*minimizer*. Finally, if u is a $(K,\{0\},\{0\})$-minimizer, we will simply say that u is a *minimizer for* K.

Remark 2.1. The local analogue for Definition 2.4 only requires (4) to hold for $\mathcal{B}_{y,\rho} \Subset \Omega$.

We will assume that the integrand for the functional behaves like a radial function g when the modulus of the argument is sufficiently large. More precisely, we have the following definition.

Definition 2.5. Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}^N$ and $g : [0,\infty) \to [0,\infty)$ be given. We will say that f is *asymptotically related to* g if for every $\varepsilon > 0$ there is a $\sigma_\varepsilon \geqslant 1$ such that

$$\big|f(F) - g\big(|F|\big)\big| < \varepsilon g\big(|F|\big)$$

whenever $F \in \mathbb{R}^{N\times n}$ satisfies $|F| > \sigma_\varepsilon$.

We will impose the following structure on the function g.

Definition 2.6. If a function $g : [0,\infty) \to [0,\infty)$ satisfies

$$\begin{cases} \text{(i)}\ \ g \in W^{2,1}_{\mathrm{loc}}([0,\infty)) \text{ is convex and } g(0) = g'(0) = 0, \\ \text{(ii)}\ \ (p-1)\frac{g'(t)}{t} \leqslant g''(t) \leqslant (q-1)\frac{g'(t)}{t} \text{ for a.e. } t > 0, \\ \text{(iii)}\ \ g(1) > 0 \end{cases} \tag{5}$$

for some $1 < p \leqslant q < \infty$, then we will say that g has (p,q)*-structure*.

Remark 2.2. Owing to Lemma 3.1(iii), we see that condition (iii) is equivalent to the condition $g(t_0) > 0$ for some $t_0 > 0$, which is in turn equivalent to the condition $g(t) > 0$ for all $t > 0$. We also note that conditions (i) and (ii) imply that $g \in W^{2,\infty}_{\mathrm{loc}}((0,\infty))$, but not necessarily that $g \in W^{2,\infty}_{\mathrm{loc}}([0,\infty))$, since g'' may be unbounded near the origin if $p < 2$.

Eventually, we will restrict ourselves to families of functions $\{g_y\}_{y\in\mathcal{U}}$ that satisfy the conditions for Definition 2.6 in a uniform sense.

Definition 2.7. Suppose that $\mathcal{U} \subset \mathbb{R}^n$ and that $\{g_y\}_{y\in\mathcal{U}}$ is a family of functions that each have (p,q)-structure, with $1 < p \leqslant q$. If there is a finite constant $c \geqslant 1$ and a function g having (p,q)-structure such that $g(t) \leqslant g_y(t) \leqslant cg(t)$ for all $t \geqslant 1$, then we say that the family $\{g_y\}_{y\in\mathcal{U}}$ has *uniform* (p,q)*-structure*.

Definition 2.8. Let $\mathcal{U} \subset \mathbb{R}^n$ be measurable. Suppose that $\{f_y\}_{y\in\mathcal{U}}$ is a family of functions defined on $\mathbb{R}^{N\times n}$ and that $\{g_y\}_{y\in\mathcal{U}}$ is a family of functions with uniform (p,q)-structure. With the function g given in Definition 2.7, we say that the two families are $L^{g,\kappa}$*-asymptotically related*, if for every $\varepsilon > 0$ there is a $\sigma_\varepsilon \in L^{g,\kappa}(\mathcal{U})$ such that for every $y \in \mathcal{U}$, the inequality

$$\big|f_y(F) - g_y\big(|F|\big)\big| < \varepsilon g_y\big(|F|\big)$$

holds whenever $|F| > \sigma_\varepsilon(y)$.

Remark 2.3. The local analogue of the above definition is defined in the obvious way.

3. Preliminary lemmas

At this point, we present several lemmas that are used throughout the paper.

Lemma 3.1. *Let* $g : [0,\infty) \to [0,\infty)$ *be a function satisfying* (5)(i) *and* (5)(ii). *Then the following hold*:

(i) $pg(t) \leqslant tg'(t) \leqslant qg(t)$ *for every* $t > 0$,
(ii) $p(p-1)g(t) \leqslant t^2 g''(t) \leqslant q(q-1)g(t)$ *for a.e.* $t > 0$,
(iii) $c^p g(t) \leqslant g(ct) \leqslant c^q g(t)$ *for every* $t > 0$ *and* $c \geqslant 1$,
(iv) $g(1)(t^p - 1) \leqslant g(t) \leqslant g(1)(t^q + 1)$ *for every* $t > 0$,
(v) $g(s+t) \leqslant 2^q(g(s) + g(t))$ *for every* $s, t \geqslant 0$, *and*
(vi) $tg'(s) \leqslant g(t) + (q-1)g(s)$ *for every* $s, t \geqslant 0$.

Proof. First we prove part (i). Since $g(0) = 0$, we write $g(t) = \int_0^t g'(s)\,\mathrm{d}s$. The left side of (5)(ii) and integration by parts yields $g(t) \leqslant \frac{1}{p-1}[tg'(t) - g(t)]$. Solving the inequality for $tg'(t)$, we get $pg(t) \leqslant tg'(t)$. A similar argument proves $tg'(t) \leqslant qg(t)$. Part (ii) follows immediately from (i) and (5)(ii). For part (iii), we first observe that either g is identically zero, or g is a positive Young function. Thus (iii) is an immediate consequence of part (i) and Propositions 2.1 and 2.3 in [3]. For (iv), note that the result is obvious if $0 \leqslant t < 1$. If $t \geqslant 1$, we use part (iii) to get $g(1)t^p \leqslant g(t) \leqslant g(1)t^q$. To prove (v), we can assume without loss of generality that $s \leqslant t$. Then $s + t \leqslant 2t$, so part (iii) gives the result. Finally, we turn our attention to part (vi). Let $g^* : [0,\infty) \to [0,\infty)$ denote the Young conjugate function of g, defined by

$$g^*(\tau) := \sup_{\sigma \in [0,\infty)} \{\sigma\tau - g(\sigma)\}.$$

Using the facts that $g \in \mathcal{C}^1([0,\infty))$ and that g' is strictly increasing, we find that $g^*(\tau) = \tau(g')^{-1}(\tau) - g((g')^{-1}(\tau))$. We also clearly have $\sigma\tau \leqslant g(\sigma) + g^*(\tau)$ for every σ, $\tau \in [0,\infty)$. Using these facts along with part (i), we obtain

$$tg'(s) \leqslant g(t) + g^*\big(g'(s)\big) = g(t) + sg'(s) - g(s) \leqslant g(t) + (q-1)g(s),$$

which finishes the proof. □

Lemma 3.2. *Suppose that* $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ *is asymptotically related to* g. *If* f *satisfies the growth condition*

$$\big|f(F)\big| \leqslant Lg\big(|F|\big) + \alpha$$

for some positive constants L *and* α, *then*

$$g\big(|F|\big) \leqslant 2f(F) + c$$

for all $F \in \mathbb{R}^{N\times n}$, *where* $c := (2L+1)g(\sigma_{1/2}) + 2\alpha$.

Proof. If $|F| \leqslant \sigma_{1/2}$, then $g(|F|) \leqslant g(\sigma_{1/2})$. Therefore, since f satisfies the growth condition, we must have $g(|F|) \leqslant 2f(F) + c$, where c is as in the statement of the lemma. On the other hand, if $|F| > \sigma_{1/2}$, then since f is asymptotically related to g, we deduce that $g(|F|) - f(F) \leqslant \frac{1}{2}g(|F|)$, from which the result follows. □

Lemma 3.3. *Suppose that* g *is a function with* (p,q)*-structure, and let* $\alpha \geqslant 0$. *Then there is a positive constant* $C = C(p,q)$, *independent of* α, *such that*

$$\left[\int_0^t s^\alpha g'(s)^\alpha \sqrt{g''(s)}\,\mathrm{d}s\right]^2 \geqslant \frac{1}{C(2\alpha+1)^2} t^{2\alpha+1} g'(t)^{2\alpha+1}.$$

Proof. Using (5)(ii), integrating by parts, then using (5)(ii) again, we obtain

$$\int_0^t s^\alpha g'(s)^\alpha \sqrt{g''(s)}\,\mathrm{d}s \geqslant \frac{2\sqrt{p-1}}{2\alpha+1} t^{\alpha+\frac{1}{2}} g'(t)^{\alpha+\frac{1}{2}} - \sqrt{(p-1)(q-1)} \int_0^t s^\alpha g'(s)^\alpha \sqrt{g''(s)}\,\mathrm{d}s.$$

Solving the inequality for the integral and squaring both sides yields the result. $\square$

The next lemma is essentially a restatement of Lemma 1 in [9], and is proved there.

Lemma 3.4. *Let $\varphi : (0,\infty) \to \mathbb{R}$ be given, and suppose that there exist $A \geqslant 1$, $R_0 > 0$, and $\alpha > \beta \geqslant 0$ such that for some $0 \leqslant \varepsilon \leqslant (\frac{1}{2A})^{\frac{2\alpha}{\alpha-\beta}}$, the inequality*

$$\varphi(\rho) \leqslant A\left[\left(\frac{\rho}{R}\right)^\alpha + \varepsilon\right]\varphi(R) + A\frac{R^{\alpha+\beta}}{\rho^\alpha}$$

holds for each $0 < \rho \leqslant R \leqslant R_0$. Then there is some finite constant $B = B(A,\alpha,\beta)$ such that

$$\varphi(\rho) \leqslant B\left(\frac{\rho}{R}\right)^\beta \varphi(R) + B\rho^\beta$$

for all $0 < \rho \leqslant R \leqslant R_0$.

The following lemma establishes that the Euler–Lagrange equations hold in the weak sense for minimizers of functionals having integrands with (p,q)-structure. It can be proved using the same strategy that Evans uses to prove Theorem 4 on page 451 in [8]. The main modification required in the proof is to use the Young conjugate function of g (see the proof of Lemma 3.1) instead of Young's inequality.

Lemma 3.5. *Let g be a function with (p,q)-structure, and let $G_0 \in \mathbb{R}^{n\times n}$ be invertible. If v is a minimizer of $I(w) := \int_\Omega g(|\nabla w G_0|)\,\mathrm{d}x$, then*

$$\int_\Omega \frac{\partial}{\partial F} g\big(|\nabla v G_0|\big) : \nabla\varphi G_0 \,\mathrm{d}x = 0$$

for every $\varphi \in W_0^{1,1}(\Omega;\mathbb{R}^N)$ satisfying $I(\varphi) < \infty$.

4. Lipschitz regularity results

In this section, we prove a refinement of the local Lipschitz regularity result established in [16]. As discussed in the introduction, our strategy is very similar to the one employed in [16]. We consider the functional

$$J(v) := \int_\Omega g\big(|\nabla v|\big)\,\mathrm{d}x, \tag{6}$$

where g has (p,q)-structure. We temporarily make the assumption that there are positive constants μ and M such that for all $\xi, \lambda \in \mathbb{R}^{N\times n}$, the following holds:

$$\mu|\lambda|^2 \leqslant \sum_{i,j,\alpha,\beta} \frac{\partial^2}{\partial\xi_i^\alpha \partial\xi_j^\beta} g\big(|\xi|\big)\lambda_i^\alpha \lambda_j^\beta \leqslant M|\lambda|^2. \tag{7}$$

One can show (see the first paragraph of Section 3 in [16]) that

$$|\lambda|^2 \min\left\{g''(|\xi|), \frac{g'(|\xi|)}{|\xi|}\right\} \leqslant \sum_{i,j,\alpha,\beta} \frac{\partial^2}{\partial\xi_i^\alpha \partial\xi_j^\beta} g(|\xi|)\lambda_i^\alpha \lambda_j^\beta \leqslant |\lambda|^2 \max\left\{g''(|\xi|), \frac{g'(|\xi|)}{|\xi|}\right\}$$

for all $\lambda, \xi \in \mathbb{R}^{N\times n}$. Therefore (7) is satisfied if both $g''(|\xi|)$ and $g'(|\xi|)/|\xi|$ are bounded below by μ and above by M. Assumption (7) gives quadratic growth of g, which in turn forces any minimizer to be of class $W^{2,2} \cap W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$.

The following lemma provides an estimate of the form (3) for minimizers of (6) under the additional assumptions that (7) holds and that g'' is continuous. The estimate we obtain is independent of the constants μ and M, which allows us to eventually remove both the assumption in (7) and the continuity assumption on g''.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^n$ be open, and let $g \in \mathcal{C}^2([0,\infty))$ be a function with (p,q)-structure that satisfies* (7). *Suppose that $v \in W^{1,1}(\Omega; \mathbb{R}^N)$ is a minimizer of* (6). *Then there is a constant $C = C(n,p,q)$ such that*

$$\left\| g(|\nabla v|)\right\|_{L^\infty(\mathcal{B}_{x_0,\rho})} \leqslant \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{x_0,R}} g(|\nabla v|)\,\mathrm{d}x$$

whenever $\mathcal{B}_{x_0,R} \subset \Omega$ and $0 < \rho < R$.

Proof. First, we establish that

$$\left\| g(|\nabla v|)\right\|_{L^\infty(\mathcal{B}_{x_0,\frac{r}{2}})} \leqslant \frac{C}{r^n} \int_{\mathcal{B}_{x_0,r}} g(|\nabla v|)\,\mathrm{d}x \tag{8}$$

for any $x_0 \in \Omega$ and $r > 0$ such that $B_{x_0,r} \subset \Omega$. Using a rescaling argument, we see that it suffices to show (8) when $x_0 = 0$ and $r = 1$. Following the first part of the proof for Lemma 4.1 in [16], it can be shown that $v \in W^{2,2}(\mathcal{B}) \cap W^{1,\infty}_{\mathrm{loc}}(\mathcal{B})$ and

$$\int_{\mathcal{B}} \eta^2 \Phi(|\nabla v|) g''(|\nabla v|)\left|\nabla(|\nabla v|)\right|^2 \mathrm{d}x \leqslant C \int_{\mathcal{B}} \Phi(|\nabla v|) g''(|\nabla v|)|\nabla\eta|^2 |\nabla v|^2 \,\mathrm{d}x$$

for every $\eta \in \mathcal{C}^1_c(\mathcal{B})$ and Φ that is nondecreasing, continuous on $[0,\infty)$, and Lipschitz continuous on $[\varepsilon, T]$ for all $T > \varepsilon > 0$. Thus, for fixed $\alpha \geqslant 0$, we can define $\Phi(t) = t^{2\alpha} g'(t)^{2\alpha}$; with this definition of Φ, the above inequality becomes

$$\begin{aligned} &\int_{\mathcal{B}} \eta^2 |\nabla v|^{2\alpha} g'(|\nabla v|)^{2\alpha} g''(|\nabla v|)\left|\nabla(|\nabla v|)\right|^2 \mathrm{d}x \\ &\quad \leqslant C \int_{\mathcal{B}} |\nabla v|^{2\alpha+2} g'(|\nabla v|)^{2\alpha} g''(|\nabla v|)|\nabla\eta|^2 \,\mathrm{d}x. \end{aligned}$$

Using (5)(ii) in this inequality, we get

$$\begin{aligned} &\int_{\mathcal{B}} \eta^2 |\nabla v|^{2\alpha} g'(|\nabla v|)^{2\alpha} g''(|\nabla v|)\left|\nabla(|\nabla v|)\right|^2 \mathrm{d}x \\ &\quad \leqslant C \int_{\mathcal{B}} |\nabla v|^{2\alpha+1} g'(|\nabla v|)^{2\alpha+1} |\nabla\eta|^2 \,\mathrm{d}x. \end{aligned} \tag{9}$$

Now define $G:[0,\infty)\to[0,\infty)$ by $G(t):=\int_0^t s^\alpha g'(s)^\alpha\sqrt{g''(s)}\,\mathrm{d}s$. Since g' is increasing and g satisfies (5)(ii), by Hölder's inequality we obtain

$$\big[G(t)\big]^2\leqslant t^{2\alpha+1}g'(t)^{2\alpha}\int_0^t g''(s)\,\mathrm{d}s=t^{2\alpha+1}g'(t)^{2\alpha+1}.$$

Hence we see that

$$\begin{aligned}\big|\nabla\big(\eta G(|\nabla v|)\big)\big|^2&=\big|(\nabla\eta)G\big(|\nabla v|\big)+\eta G'\big(|\nabla v|\big)\nabla\big(|\nabla v|\big)\big|^2\\&\leqslant 2|\nabla\eta|^2|\nabla v|^{2\alpha+1}g'\big(|\nabla v|\big)^{2\alpha+1}\\&\quad+2\eta^2|\nabla v|^{2\alpha}g'\big(|\nabla v|\big)^{2\alpha}g''\big(|\nabla v|\big)\big|\nabla\big(|\nabla v|\big)\big|^2.\end{aligned}$$

Note that the assumption in (7) implies that ∇v is locally bounded. Integrating the above inequality over $\mathcal{B}$, using (9) and Sobolev's inequality, we deduce that there is a constant C depending only upon n such that

$$\left\{\int_{\mathcal{B}}\eta^{2^*}\big[G\big(|\nabla v|\big)^2\big]^{\frac{2^*}{2}}\,\mathrm{d}x\right\}^{\frac{2}{2^*}}\leqslant C\int_{\mathcal{B}}|\nabla\eta|^2\big[|\nabla v|g'\big(|\nabla v|\big)\big]^{2\alpha+1}\,\mathrm{d}x. \tag{10}$$

If $n=2$, we select 2^* to be any finite number strictly larger than 2. Recalling the definition of G and using Lemma 3.3, from (10) we obtain

$$\left\{\int_{\mathcal{B}}\eta^{2^*}\big[|\nabla v|g'\big(|\nabla v|\big)\big]^{\frac{2^*}{2}(2\alpha+1)}\,\mathrm{d}x\right\}^{\frac{2}{2^*}}\leqslant C(2\alpha+1)^2\int_{\mathcal{B}}|\nabla\eta|^2\big[|\nabla v|g'\big(|\nabla v|\big)\big]^{2\alpha+1}\,\mathrm{d}x. \tag{11}$$

Now let $0<\rho<R\leqslant 1$ be given, and let η be a non-negative test function that is equal to 1 in $\mathcal{B}_\rho$, has support contained in $\mathcal{B}_R$, and is such that $|\nabla\eta|\leqslant\frac{C}{R-\rho}$; then from (11), we see that

$$\left\{\int_{\mathcal{B}_\rho}\big[|\nabla v|g'\big(|\nabla v|\big)\big]^{\frac{2^*}{2}(2\alpha+1)}\,\mathrm{d}x\right\}^{\frac{2}{2^*}}\leqslant\frac{C(2\alpha+1)^2}{(R-\rho)^2}\int_{\mathcal{B}_R}\big[|\nabla v|g'\big(|\nabla v|\big)\big]^{2\alpha+1}\,\mathrm{d}x.$$

Now putting $\beta=2\alpha+1$ (note that $\beta\geqslant 1$, since $\alpha\geqslant 0$), we can rewrite the above inequality as

$$\left\{\int_{\mathcal{B}_\rho}\big[|\nabla v|g'\big(|\nabla v|\big)\big]^{\frac{2^*}{2}\beta}\,\mathrm{d}x\right\}^{\frac{2}{2^*}}\leqslant\frac{C\beta^2}{(R-\rho)^2}\int_{\mathcal{B}_R}\big[|\nabla v|g'\big(|\nabla v|\big)\big]^{\beta}\,\mathrm{d}x. \tag{12}$$

Define the decreasing sequence $\{\rho_i\}_{i=0}^\infty$ by $\rho_i=\frac{1}{2}(1+2^{-i})$. Then $\rho_0=1$ and ρ_i decreases to $\frac{1}{2}$ as $i\to\infty$. Also define an increasing sequence $\{\beta_i\}_{i=0}^\infty$ by $\beta_i=(\frac{2^*}{2})^i$. Thus we can rewrite (12) with $R=\rho_i$, $\rho=\rho_{i+1}$, and $\beta=\beta_i$. Upon iterating the result and substituting in the expression for β_i, we obtain

$$\left\{\int_{\mathcal{B}_{\rho_{i+1}}}\big[|\nabla v|g'\big(|\nabla v|\big)\big]^{(\frac{2^*}{2})^{i+1}}\,\mathrm{d}x\right\}^{(\frac{2}{2^*})^{i+1}}\leqslant\prod_{k=0}^{i}\big[C(2^*)^{2k}\big]^{(\frac{2}{2^*})^k}\int_{\mathcal{B}_1}|\nabla v|g'\big(|\nabla v|\big)\,\mathrm{d}x. \tag{13}$$

Now we verify that the product occurring in the above inequality remains bounded. For each i, put $A_i := \prod_{k=0}^{i} C^{(\frac{2}{2^*})^k}$ and $B_i := \prod_{k=0}^{i} (2^*)^{2k(\frac{2}{2^*})^k}$. We will estimate A_i and B_i separately. If $n \geqslant 3$, then we can bound A_i as follows:

$$A_i \leqslant \prod_{k=0}^{\infty} C^{(\frac{2}{2^*})^k} = C^{\sum_{k=0}^{\infty}(\frac{2}{2^*})^k} = C^{\frac{n}{2}}.$$

Similarly, if $n \geqslant 3$, we get that

$$B_i \leqslant (2^*)^{\frac{n(n-2)}{2}} = \left(\frac{2n}{n-2}\right)^{\frac{n(n-2)}{2}}.$$

If $n = 2$, then 2^* is a fixed number larger than 2 and we obtain similar estimates for A_i and B_i. Introducing the estimates for A_i and B_i into (13), we find that

$$\left\{ \int_{\mathcal{B}_{\rho_{i+1}}} \left[|\nabla v| g'\big(|\nabla v|\big)\right]^{(\frac{2^*}{2})^{i+1}} \mathrm{d}x \right\}^{(\frac{2}{2^*})^{i+1}} \leqslant C \int_{\mathcal{B}_1} |\nabla v| g'\big(|\nabla v|\big)\,\mathrm{d}x. \tag{14}$$

Taking the limit as $i \to \infty$ in (14) yields

$$\begin{aligned} \big\| |\nabla v| g'\big(|\nabla v|\big)\big\|_{L^\infty(\mathcal{B}_{\frac{1}{2}})} &\leqslant \lim_{i\to\infty} \left\{ \int_{\mathcal{B}_{\rho_{i+1}}} \left[|\nabla v| g'\big(|\nabla v|\big)\right]^{(\frac{2^*}{2})^{i+1}} \mathrm{d}x \right\}^{(\frac{2}{2^*})^{i+1}} \\ &\leqslant C \int_{\mathcal{B}} |\nabla v| g'\big(|\nabla v|\big)\,\mathrm{d}x. \end{aligned}$$

Using Lemma 3.1(i) in both sides of the above inequality gives

$$\big\| g\big(|\nabla v|\big)\big\|_{L^\infty(\mathcal{B}_{\frac{1}{2}})} \leqslant C \int_{\mathcal{B}_1} g\big(|\nabla v|\big)\,\mathrm{d}x. \tag{15}$$

As was mentioned at the beginning of the proof, using a rescaling argument and (15), we obtain (8).

Now we use (8) to finish the proof. Fix $0 < \rho < R$ and $x_0 \in \Omega$ satisfying $B_{x_0,R} \subset \Omega$, and let $y \in \mathcal{B}_{x_0,\rho}$. Then $\mathcal{B}_{y,R-\rho} \subset \mathcal{B}_{x_0,R}$, and so taking $r = R - \rho$ in (8) yields

$$\big\| g\big(|\nabla v|\big)\big\|_{L^\infty(\mathcal{B}_{y,\frac{R-\rho}{2}})} \leqslant \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{y,R-\rho}} g\big(|\nabla v|\big)\,\mathrm{d}x \leqslant \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{x_0,R}} g\big(|\nabla v|\big)\,\mathrm{d}x.$$

Since the above inequality holds for all $y \in \mathcal{B}_{x_0,\rho}$, we conclude that

$$\big\| g\big(|\nabla v|\big)\big\|_{L^\infty(\mathcal{B}_{x_0,\rho})} \leqslant \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{x_0,R}} g\big(|\nabla v|\big)\,\mathrm{d}x,$$

which was to be shown. $\square$

Now we will assume that g has (p,q)-structure, but does not necessarily satisfy (7), and also is not necessarily of class C^2. Our strategy is the same as that in [16]. We define a sequence of functions $\{g_k\}_{k=1}^{\infty}$ that approximate g and satisfy (7); we also define a corresponding sequence of integral functionals $\{J_k\}_{k=1}^{\infty}$. The conclusion of Lemma 4.1 holds for minimizers of J_k; we show that we can pass to the limit to get the result for the minimizer of the original functional.

Since we are assuming g has (p,q)-structure, by Remark 2.2 we see that $g(t) > 0$ for all positive t. Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a sequence of positive numbers decreasing to 0, choosing $\varepsilon_1 < 1$ sufficiently small so that $g'(\frac{1}{\varepsilon_1}) \geqslant 1$. We define $g_k' : [0,+\infty) \to [0,+\infty)$ by

$$g_k'(t) = \begin{cases} \frac{g'(\varepsilon_k)}{\varepsilon_k} t, & 0 \leqslant t \leqslant \varepsilon_k, \\ g'(t), & \varepsilon_k < t \leqslant \frac{1}{\varepsilon_k}, \\ \min\{\varepsilon_k g'(\frac{1}{\varepsilon_k})t, g'(t) + \varepsilon_k t - 1\}, & t > \frac{1}{\varepsilon_k}. \end{cases} \tag{16}$$

Now we define g_k as

$$g_k(t) = \int_0^t g_k'(s)\,\mathrm{d}s, \tag{17}$$

where g_k' is defined in (16). Then $g_k \in W^{2,\infty}_{\mathrm{loc}}([0,\infty))$ and satisfies (5)(i) and (7) for some positive constants μ_k and M_k. We compute g_k for $t \leqslant \frac{1}{\varepsilon_k}$, and find that

$$g_k(t) = \begin{cases} \frac{g'(\varepsilon_k)}{2\varepsilon_k} t^2, & 0 \leqslant t \leqslant \varepsilon_k, \\ g(t) + \frac{g'(\varepsilon_k)(\varepsilon_k)}{2} - g(\varepsilon_k), & \varepsilon_k \leqslant t \leqslant \frac{1}{\varepsilon_k}. \end{cases} \tag{18}$$

For the remainder of the section, g will be a function with (p,q)-structure and g_k will be the approximating functions defined in (17).

Lemma 4.2. *Fix $k \in \mathbb{N}$, and assume that $v \in W^{1,1}(\Omega;\mathbb{R}^N)$ is a minimizer for the functional*

$$u \mapsto \int_\Omega g_k\big(|\nabla u|\big)\,\mathrm{d}x.$$

Then there is a constant $C = C(n,p,q)$ such that

$$\big\|g_k\big(|\nabla v|\big)\big\|_{L^\infty(\mathcal{B}_{x_0,\rho})} \leqslant \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{x_0,R}} g_k\big(|\nabla v|\big)\,\mathrm{d}x$$

whenever $\mathcal{B}_{x_0,R} \subset \Omega$ and $0 < \rho < R$.

Proof. Note that g_k is only of class $W^{2,\infty}_{\mathrm{loc}}$, so we may not simply apply Lemma 4.1, which would require g_k to be $\mathcal{C}^2$. Our strategy is to mollify g_k, apply Lemma 4.1 to the minimizers of the functionals involving the mollifications of g_k, then pass to the limit to obtain the result for the original minimizer. Before we perform the mollification, let us extend g_k to an even function on all of $\mathbb{R}$. Now, for every $0 < \delta < \varepsilon_k^2/4$, let g_k^δ denote a standard mollification of g_k, where the support of the mollifier is contained in $[-\delta,\delta]$. Then $(g_k^\delta)'(0) = 0$, but $g_k^\delta(0) > 0$. Define $g_\delta : [0,\infty) \to [0,\infty)$ by

$$g_\delta(t) := g_k^\delta(t) - g_k^\delta(0).$$

Then (5)(i) holds for g_δ. Recall that $\{\varepsilon_k\}_{k=1}^{\infty}$ was chosen to be a decreasing sequence with $\varepsilon_1 < 1$ selected small enough so that $g'(1/\varepsilon_k) \geqslant 1$; keeping this in mind, it is straightforward to show that g_k has $(\bar{p}, \bar{q})$-structure, where $\bar{p} := \min\{p, 2\}$ and $\bar{q} := \max\{q+1, 3\}$. Using this and the fact that $\delta < 1/4$, we can show that g_δ has $(\tilde{p}, \tilde{q})$-structure, where we have put $\tilde{p} := \min\{\frac{5}{3}, \frac{1}{3} + \frac{2p}{3}\}$ and $\tilde{q} := \max\{2q+1, 5\}$. We also find that g_δ satisfies (7) for the same μ_k, M_k as g_k. Suppose $\mathcal{B}_{x_0,R} \subset \Omega$, and let $v_\delta \in W^{1,1}(\Omega; \mathbb{R}^N)$ be a minimizer of the functional

$$u \mapsto \int_{\mathcal{B}_{x_0,R}} g_\delta\big(|\nabla u|\big)\,\mathrm{d}x$$

satisfying $v_\delta = v$ on $\partial\mathcal{B}_{x_0,R}$. Using Lemma 4.1 and the minimality of v_δ, we obtain

$$\big\| g_\delta\big(|\nabla v_\delta|\big)\big\|_{L^\infty(\mathcal{B}_{x_0,\rho})} \leqslant \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{x_0,R}} g_\delta\big(|\nabla v_\delta|\big)\,\mathrm{d}x \leqslant \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{x_0,R}} g_\delta\big(|\nabla v|\big)\,\mathrm{d}x \tag{19}$$

for every $0 < \rho < R$. Using the convexity of g_k, it is not difficult to see that

$$g_k(t) - g_k^\delta(0) \leqslant g_\delta(t) \leqslant g_k(t+\delta) + g_k^\delta(0), \tag{20}$$

for all $t \geqslant 0$. Using (20) in (19), we find that

$$\big\| g_k\big(|\nabla v_\delta|\big)\big\|_{L^\infty(\mathcal{B}_{x_0,\rho})} \leqslant \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{x_0,R}} \big\{g_k\big(|\nabla v| + \delta\big) + g_k^\delta(0)\big\}\,\mathrm{d}x + g_k^\delta(0) \leqslant c_1, \tag{21}$$

where c_1 depends on n, p, q, k, ρ, and R. Hence $g(|\nabla v_\delta|)$ is equibounded with respect to δ in $\mathcal{B}_{x_0,\rho}$. Using Lemma 3.1(iv), we deduce that $\||\nabla v_\delta|\|_{L^\infty(\mathcal{B}_{x_0,\rho})}$ is equibounded, and so, up to a subsequence, ∇v_δ converges to some ∇w in the weak* topology of $L^\infty(\mathcal{B}_{x_0,\rho}; \mathbb{R}^{N\times n})$ for every $\rho < R$. Passing to the limit in (21), we obtain

$$\big\| g_k\big(|\nabla w|\big)\big\|_{L^\infty(\mathcal{B}_{x_0,\rho})} \leqslant \liminf_{\delta\to 0^+} \big\| g_k\big(|\nabla v_\delta|\big)\big\|_{L^\infty(\mathcal{B}_{x_0,\rho})} \leqslant \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{x_0,R}} g_k\big(|\nabla v|\big)\,\mathrm{d}x. \tag{22}$$

Using (20), the minimality of v_δ, and the dominated convergence theorem, we estimate that

$$\begin{aligned} \limsup_{\delta\to 0^+} \int_{\mathcal{B}_{x_0,R}} g_k\big(|\nabla v_\delta|\big)\,\mathrm{d}x &\leqslant \limsup_{\delta\to 0^+} \int_{\mathcal{B}_{x_0,R}} g_\delta\big(|\nabla v_\delta|\big)\,\mathrm{d}x \\ &\leqslant \lim_{\delta\to 0^+} \int_{\mathcal{B}_{x_0,R}} g_\delta\big(|\nabla v|\big)\,\mathrm{d}x = \int_{\mathcal{B}_{x_0,R}} g_k\big(|\nabla v|\big)\,\mathrm{d}x. \end{aligned} \tag{23}$$

Lemma 3.1(iv) and (23) imply that $\|\nabla v_\delta\|_{L^{\bar{p}}(\mathcal{B}_{x_0,R})}$ is uniformly bounded, so ∇v_δ converges in the weak topology of $L^{\bar{p}}(\mathcal{B}_{x_0,R}; \mathbb{R}^{N\times n})$ to ∇w. Therefore by weak lower semicontinuity and (23), we have

$$\int_{B_{x_0,R}} g_k\big(|\nabla w|\big)\,\mathrm{d}x \leqslant \liminf_{\delta\to 0^+} \int_{\mathcal{B}_{x_0,R}} g_k\big(|\nabla v_\delta|\big)\,\mathrm{d}x \leqslant \int_{\mathcal{B}_{x_0,R}} g_k\big(|\nabla v|\big)\,\mathrm{d}x.$$

Hence w is also a minimizer for the functional $u \mapsto \int_\Omega g_k(|\nabla u|)\,dx$. Since $g_k(|\cdot|)$ is strictly convex on $\mathbb{R}^{N\times n}$, the minimizer for the Dirichlet problem is unique, and so $w = v$. Therefore we can replace w with v in (22) and obtain the result. □

Lemma 4.3. *There are decreasing sequences $\{\beta_k\}_{k=1}^\infty$ and $\{\gamma_k\}_{k=1}^\infty$ converging to 0 such that $g_k(t) \leqslant g(t) + \beta_k t^2 + \gamma_k$ for all $t \geqslant 0$ and $k \in \mathbb{N}$.*

Proof. If $0 \leqslant t \leqslant \frac{1}{\varepsilon_k}$, then we can use (18) to get $g_k(t) \leqslant g(t) + \frac{1}{2}\varepsilon_k g'(\varepsilon_k)$. If $t > \frac{1}{\varepsilon_k}$, then $g_k'(s) \leqslant g'(s) + \varepsilon_k s$ for all $s > \frac{1}{\varepsilon_k}$, so by (18) we have

$$g_k(t) = g_k\left(\frac{1}{\varepsilon_k}\right) + \int_{\frac{1}{\varepsilon_k}}^{t} g_k'(s)\,ds \leqslant g_k\left(\frac{1}{\varepsilon_k}\right) + \int_{\frac{1}{\varepsilon_k}}^{t} \big(g'(s) + \varepsilon_k s\big)\,ds$$
$$\leqslant g(t) + \frac{1}{2}\varepsilon_k t^2 + \frac{1}{2}\varepsilon_k g'(\varepsilon_k).$$

We see that the lemma is proved upon taking $\beta_k = \frac{1}{2}\varepsilon_k$ and $\gamma_k = \frac{1}{2}\varepsilon_k g'(\varepsilon_k)$. □

Equipped with these lemmas, we can prove the following theorem.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be open and g be a function with (p,q)-structure. Suppose that $v \in W^{1,1}(\Omega;\mathbb{R}^N)$ is a minimizer for the functional in (6). Then there is a constant $C = C(n,p,q)$ such that*

$$\big\|g\big(|\nabla v|\big)\big\|_{L^\infty(\mathcal{B}_{x_0,\rho})} \leqslant \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{x_0,R}} g\big(|\nabla v|\big)\,dx$$

whenever $\mathcal{B}_{x_0,R} \subset \Omega$ and $0 < \rho < R$.

Proof. First assume that $\mathcal{B}_{x_0,2R} \subset \Omega$. For each $k \in \mathbb{N}$, define the integral functional

$$J_k(u) = \int_{\mathcal{B}_{x_0,R}} g_k\big(|\nabla u|\big)\,dx,$$

where g_k is as defined in (17). For each $0 < \sigma < \min\{1,R\}$, let v_σ be a smooth function defined from v using a standard mollifier. Then we have that $v_\sigma \in W^{1,2}(\mathcal{B}_{x_0,R};\mathbb{R}^N)$. Let $v_{k,\sigma}$ be a minimizer of J_k that satisfies $v_{k,\sigma} = v_\sigma$ on $\partial\mathcal{B}_{x_0,R}$. Then by Lemma 4.2, there is a constant C, independent of k and σ, such that

$$\big\|g_k\big(|\nabla v_{k,\sigma}|\big)\big\|_{L^\infty(\mathcal{B}_{x_0,\rho})} \leqslant \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{x_0,R}} g_k\big(|\nabla v_{k,\sigma}|\big)\,dx. \tag{24}$$

Since $v_{k,\sigma}$ is a minimizer for J_k, we have that

$$\int_{\mathcal{B}_{x_0,R}} g_k\big(|\nabla v_{k,\sigma}|\big)\,dx \leqslant \int_{\mathcal{B}_{x_0,R}} g_k\big(|\nabla v_\sigma|\big)\,dx. \tag{25}$$

By Lemma 4.3, we obtain decreasing sequences $\{\beta_k\}_{k=1}^{\infty}$ and $\{\gamma_k\}_{k=1}^{\infty}$ converging to 0 such that

$$\int_{\mathcal{B}_{x_0,R}} g_k\big(|\nabla v_\sigma|\big)\,\mathrm{d}x \leqslant \int_{\mathcal{B}_{x_0,R}} \big\{g\big(|\nabla v_\sigma|\big) + \beta_k|\nabla v_\sigma|^2 + \gamma_k\big\}\,\mathrm{d}x. \tag{26}$$

By properties of mollifiers,

$$\int_{\mathcal{B}_{x_0,R}} \big\{g\big(|\nabla v_\sigma|\big) + \beta_k|\nabla v_\sigma|^2 + \gamma_k\big\}\,\mathrm{d}x \leqslant \int_{\mathcal{B}_{x_0,R+\sigma}} g\big(|\nabla v|\big)\,\mathrm{d}x + \int_{\mathcal{B}_{x_0,R}} \big\{\beta_k|\nabla v_\sigma|^2 + \gamma_k\big\}\,\mathrm{d}x. \tag{27}$$

Combining (24)–(27), we have

$$\begin{aligned}\big\|g_k\big(|\nabla v_{k,\sigma}|\big)\big\|_{L^\infty(B_{x_0,\rho})} &\leqslant \frac{C}{(R-\rho)^n}\bigg[\int_{\mathcal{B}_{x_0,R+\sigma}} g\big(|\nabla v|\big)\,\mathrm{d}x + \int_{\mathcal{B}_{x_0,R}} \big\{\beta_k|\nabla v_\sigma|^2 + \gamma_k\big\}\,\mathrm{d}x\bigg],\\ &\leqslant c_{1,\sigma},\end{aligned} \tag{28}$$

where, in addition to the explicit dependence on σ, $c_{1,\sigma}$ also depends on n, p, q, R, and ρ. It follows that $\||\nabla v_{k,\sigma}|\|_{L^\infty(\mathcal{B}_{x_0,\rho})}$ is uniformly bounded in k by some $M_\sigma < \infty$. Hence there is a subsequence of $v_{k,\sigma}$ that converges in the weak* topology of $W^{1,\infty}(\mathcal{B}_{x_0,\rho};\mathbb{R}^N)$ to some function w_σ. Also, since $|\nabla v_{k,\sigma}| \leqslant M_\sigma$ in $\mathcal{B}_{x_0,\rho}$, for k large enough so that $\frac{1}{\varepsilon_k} \geqslant M_\sigma$, the computation in (18) gives

$$\big\|g_k\big(|\nabla v_{k,\sigma}|\big) - g\big(|\nabla v_{k,\sigma}|\big)\big\|_{L^\infty(\mathcal{B}_{x_0,\rho})} \leqslant \frac{g'(\varepsilon_k)\varepsilon_k}{2} + g(\varepsilon_k). \tag{29}$$

Using (29) and going to the limit in (28), we obtain

$$\begin{aligned}\liminf_{k\to\infty}\big\|g\big(|\nabla v_{k,\sigma}|\big)\big\|_{L^\infty(\mathcal{B}_{x_0,\rho})} &\leqslant \liminf_{k\to\infty}\big\|g_k\big(|\nabla v_{k,\sigma}|\big)\big\|_{L^\infty(\mathcal{B}_{x_0,\rho})}\\ &\leqslant \frac{C}{(R-\rho)^n}\int_{\mathcal{B}_{1+\sigma}} g\big(|\nabla v|\big)\,\mathrm{d}x.\end{aligned} \tag{30}$$

By properties of weak* convergent sequences, we have

$$\big\|g\big(|\nabla w_\sigma|\big)\big\|_{L^\infty(\mathcal{B}_{x_0,\rho})} \leqslant \liminf_{k\to\infty}\big\|g\big(|\nabla v_{k,\sigma}|\big)\big\|_{L^\infty(\mathcal{B}_{x_0,\rho})}. \tag{31}$$

Combining (31) and (30), we get

$$\big\|g\big(|\nabla w_\sigma|\big)\big\|_{L^\infty(\mathcal{B}_{x_0,\rho})} \leqslant \frac{C}{(R-\rho)^n}\int_{\mathcal{B}_{R+\sigma}} g\big(|\nabla v|\big)\,\mathrm{d}x \leqslant c_2, \tag{32}$$

where $c_2 := \frac{C}{(R-\rho)^n}\int_{\mathcal{B}_{x_0,2R}} g(|\nabla v|)\,\mathrm{d}x$. Therefore, by Lemma 3.1(iv), we have that ∇w_σ is uniformly bounded in $L^\infty(\mathcal{B}_{x_0,\rho};\mathbb{R}^{N\times n})$, and so we can extract a subsequence that converges weak* in $L^\infty(\mathcal{B}_{x_0,\rho};\mathbb{R}^{N\times n})$ to a function ∇w for some w.

We will show that $w = v$. By lower semicontinuity, we have

$$\int_{\mathcal{B}_{x_0,\rho}} g\big(|\nabla w_\sigma|\big)\,\mathrm{d}x \leqslant \liminf_{k\to\infty}\int_{\mathcal{B}_{x_0,\rho}} g\big(|\nabla v_{k,\sigma}|\big)\,\mathrm{d}x. \tag{33}$$

Using (29), we obtain

$$\liminf_{k\to\infty} \int\limits_{\mathcal{B}_{x_0,\rho}} g\big(|\nabla v_{k,\sigma}|\big)\,\mathrm{d}x \leqslant \liminf_{k\to\infty} \int\limits_{\mathcal{B}_{x_0,\rho}} g_k\big(|\nabla v_{k,\sigma}|\big)\,\mathrm{d}x \leqslant \liminf_{k\to\infty} \int\limits_{\mathcal{B}_{x_0,R}} g_k\big(|\nabla v_{k,\sigma}|\big)\,\mathrm{d}x. \tag{34}$$

But by combining (25)–(27) and taking the limit as $k \to \infty$, we find that

$$\liminf_{k\to\infty} \int\limits_{\mathcal{B}_{x_0,R}} g_k\big(|\nabla v_{k,\sigma}|\big)\,\mathrm{d}x \leqslant \int\limits_{\mathcal{B}_{x_0,R+\sigma}} g\big(|\nabla v|\big)\,\mathrm{d}x. \tag{35}$$

Collecting the inequalities in (33)–(35), we have

$$\int\limits_{\mathcal{B}_{x_0,\rho}} g\big(|\nabla w_\sigma|\big)\,\mathrm{d}x \leqslant \int\limits_{\mathcal{B}_{x_0,R+\sigma}} g\big(|\nabla v|\big)\,\mathrm{d}x.$$

Since the inequality above holds for every $\rho < R$, we conclude that

$$\int\limits_{\mathcal{B}_{x_0,R}} g\big(|\nabla w_\sigma|\big)\,\mathrm{d}x \leqslant \int\limits_{\mathcal{B}_{x_0,R+\sigma}} g\big(|\nabla v|\big)\,\mathrm{d}x. \tag{36}$$

By lower semicontinuity and (36), we get

$$\int\limits_{\mathcal{B}_{x_0,R}} g\big(|\nabla w|\big)\,\mathrm{d}x \leqslant \liminf_{\sigma\to 0} \int\limits_{\mathcal{B}_{x_0,R}} g\big(|\nabla w_\sigma|\big)\,\mathrm{d}x \leqslant \int\limits_{\mathcal{B}_{x_0,R}} g\big(|\nabla v|\big)\,\mathrm{d}x. \tag{37}$$

Since $g''(t) > 0$ for all $t > 0$, we see that $g(|\cdot|)$ is strictly convex on $\mathbb{R}^{N\times n}$. Thus the minimizer to the Dirichlet problem is unique, and so we can conclude from (37) that $w = u$. Passing to the limit in (32) yields

$$\big\|g\big(|\nabla v|\big)\big\|_{\mathcal{B}_{x_0,\rho}} \leqslant \liminf_{\sigma\to 0}\big\|g\big(|\nabla w_\sigma|\big)\big\|_{\mathcal{B}_{x_0,\rho}} \leqslant \frac{C}{(R-\rho)^n} \int\limits_{\mathcal{B}_{x_0,R}} g\big(|\nabla v|\big)\,\mathrm{d}x.$$

Thus we have shown the result if $\mathcal{B}_{x_0,2R} \subset \Omega$. Now suppose only that $\mathcal{B}_{x_0,R} \subset \Omega$, and $0 < \rho < R$. Then $\mathcal{B}_{y,R-\rho} \subset \Omega$ for every $y \in \mathcal{B}_{x_0,\rho}$, so by the argument above, we have that

$$\big\|g\big(|\nabla v|\big)\big\|_{L^\infty(\mathcal{B}_{y,\frac{R-\rho}{4}})} \leqslant \frac{C}{(R-\rho)^n} \int\limits_{\mathcal{B}_{y,\frac{R-\rho}{2}}} g\big(|\nabla v|\big)\,\mathrm{d}x \leqslant \frac{C}{(R-\rho)^n} \int\limits_{\mathcal{B}_{x_0,R}} g\big(|\nabla v|\big)\,\mathrm{d}x.$$

Since the above inequality holds for every $y \in \mathcal{B}_{x_0,\rho}$, we see that

$$\big\|g\big(|\nabla v|\big)\big\|_{L^\infty(\mathcal{B}_{x_0,\rho})} \leqslant \frac{C}{(R-\rho)^n} \int\limits_{\mathcal{B}_{x_0,R}} g\big(|\nabla v|\big)\,\mathrm{d}x,$$

which is the desired result. $\square$

We can change variables and use Theorem 4.1 to establish the apparently more general result that follows.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be open, and let $v \in W^{1,1}(\Omega; \mathbb{R}^N)$ be a minimizer of the functional*

$$u \mapsto \int_{\Omega} g(|\nabla u G_0|)\,\mathrm{d}x,$$

where g is a function with (p,q)-structure and G_0 is an invertible $n \times n$ constant matrix. Then there is a constant $C = C(n, p, q, |G_0^{-1}|, |G_0|)$ such that

$$\big\|g(|\nabla v G_0|)\big\|_{L^\infty(\mathcal{B}_{x_0,\rho})} \leqslant \frac{C}{(R-\rho)^n} \int_{\mathcal{B}_{x_0,R}} g(|\nabla v G_0|)\,\mathrm{d}x$$

whenever $\mathcal{B}_{x_0,R} \subset \Omega$ and $0 < \rho < R$.

Using a reflection argument and Theorem 4.2, we can show the following version of the result for the half-ball.

Theorem 4.3. *Let g be a function with (p,q)-structure, and suppose that $G_0 \in \mathbb{R}^{n\times n}$ is invertible. Let $v \in W^{1,1}(\Omega; \mathbb{R}^N)$ be a minimizer of the functional*

$$u \mapsto \int_{\mathcal{B}^+} g(|\nabla u G_0|)\,\mathrm{d}x,$$

satisfying $v = 0$ on $\mathcal{B} \cap \partial\mathcal{H}^+$ in the sense of trace. Then there is a constant $C = C(n, p, q, |G_0^{-1}|, |G_0|)$ such that

$$\big\|g(|\nabla v G_0|)\big\|_{L^\infty(\mathcal{B}^+_{x_0,\rho})} \leqslant \frac{C}{(R-\rho)^n} \int_{\mathcal{B}^+_{x_0,R}} g(|\nabla v G_0|)\,\mathrm{d}x$$

for any $x_0 \in \mathcal{B}^+$ and $0 < \rho < R \leqslant 1 - |x_0|$.

5. Morrey regularity

For this section, we fix $0 \leqslant \kappa < n$ and let $\{\omega_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}^0([0,\infty))$ be a family of nondecreasing functions satisfying $\omega_\varepsilon(0) = 0$ for each $\varepsilon > 0$. The following is a technical lemma that will facilitate the proof for Lemma 5.3.

Lemma 5.1. *Suppose that $\mathcal{T} \subset \mathbb{R}^n$ is measurable, and let A and B be measurable functions mapping $\mathcal{T}$ into $\mathbb{R}^{N\times n}$. If g has (p,q)-structure, then there is a constant $C = C(n, p, q)$ such that*

$$\int_{\mathcal{T}} g'(|A|)|B-A|\,\mathrm{d}x \leqslant C \int_{\mathcal{T}} \left\{ g(|B|) - \frac{\partial}{\partial F} g(|A|) : [B-A] \right\} \mathrm{d}x.$$

Proof. Let $\mathcal{U} := \{x \in \mathcal{T}\colon\ |B(x)| \leqslant 5|A(x)|\}$, and let $\mathcal{V} := \mathcal{T} \setminus \mathcal{U}$. Using Lemma 3.1(i) and the convexity of g, we get

$$\int_{\mathcal{U}} g'(|A|)|B-A|\,\mathrm{d}x \leqslant 6 \int_{\mathcal{U}} g'(|A|)|A|\,\mathrm{d}x \leqslant C \int_{\mathcal{U}} g(|A|)\,\mathrm{d}x$$

$$\leqslant C \int_{\mathcal{U}} \left\{ g\big(|B|\big) - \frac{\partial}{\partial F} g\big(|A|\big) : [B - A] \right\} \mathrm{d}x. \tag{38}$$

Next we estimate the integral over $\mathcal{V}$. Note that for all $x \in \mathcal{V}$, we have that $|A(x)| \leqslant \frac{1}{4}|B(x) - A(x)|$, and hence for $t \in [1/2, 3/4]$, we obtain the inequality

$$\frac{1}{4}|B - A| \leqslant \big|A + t[B - A]\big| \leqslant |B - A|. \tag{39}$$

Now we will show that there is a constant C such that for all $x \in \mathcal{V}$ and a.e. $t \in [1/2, 3/4]$, we have

$$g\left(\frac{1}{4}|B - A|\right) \leqslant C \frac{\mathrm{d}^2}{\mathrm{d}t^2} g\big(\big|A + t[B - A]\big|\big). \tag{40}$$

A routine computation shows that

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\big[g\big(\big|A + t[B - A]\big|\big)\big] = {} & \frac{g'(|A + t[B - A]|)}{|A + t[B - A]|}|B - A|^2 \\ & + \frac{g''(|A + t[B - A]|)}{|A + t[B - A]|^2}\big(\big[A + t(B - A)\big] : [B - A]\big)^2 \\ & - \frac{g'(|A + t[B - A]|)}{|A + t[B - A]|^3}\big(\big[A + t(B - A)\big] : [B - A]\big)^2. \end{aligned} \tag{41}$$

To obtain (40), we need to consider two cases. First, we suppose that $1 < p < 2$. Using (39), Lemma 3.1(i), and (39) again, we obtain

$$\begin{aligned} g\left(\frac{1}{4}|B - A|\right) & \leqslant g\big(\big|A + t[B - A]\big|\big) \leqslant \frac{1}{p} \frac{g'(|A + t[B - A]|)}{|A + t[B - A]|}\big|A + t[B - A]\big|^2 \\ & \leqslant \frac{1}{p} \frac{g'(|A + t[B - A]|)}{|A + t[B - A]|}|B - A|^2. \end{aligned} \tag{42}$$

We rewrite the right side of the previous inequality as follows:

$$\begin{aligned} \frac{1}{p} \frac{g'(|A + t[B - A]|)}{|A + t[B - A]|}|B - A|^2 = {} & \frac{p - 2}{p(p - 1)} \frac{g'(|A + t[B - A]|)}{|A + t[B - A]|}|B - A|^2 \\ & + \frac{1}{p(p - 1)} \frac{g'(|A + t[B - A]|)}{|A + t[B - A]|}|B - A|^2. \end{aligned}$$

Since we are assuming for the moment that $1 < p < 2$, obviously $p - 2 < 0$; therefore, from the equality above and the Cauchy–Schwartz inequality, we have that

$$\begin{aligned} \frac{1}{p} \frac{g'(|A + t[B - A]|)}{|A + t[B - A]|}|B - A|^2 \leqslant {} & \frac{p - 2}{p(p - 1)} \frac{g'(|A + t[B - A]|)}{|A + t[B - A]|^3}\big(\big[A + t[B - A]\big] : [B - A]\big)^2 \\ & + \frac{1}{p(p - 1)} \frac{g'(|A + t[B - A]|)}{|A + t[B - A]|}|B - A|^2. \end{aligned}$$

In the right side of the previous inequality, we use (5)(ii) and the computation in (41) to get

$$\frac{1}{p} \frac{g'(|A + t[B - A]|)}{|A + t[B - A]|}|B - A|^2 \leqslant \frac{1}{p(p - 1)} \frac{\mathrm{d}^2}{\mathrm{d}t^2} g\big(\big|A + t[B - A]\big|\big).$$

Combining this with (42), we see that

$$g\left(\frac{1}{4}|B-A|\right) \leqslant \frac{1}{p(p-1)}\frac{\mathrm{d}^2}{\mathrm{d}t^2}g\big(\big|A+t[B-A]\big|\big) \tag{43}$$

for $1<p<2$.

Now assume $p \geqslant 2$. Then $g'(s)/s$ is increasing; using this fact along with Lemma 3.1(i) and (39), we find that

$$g\left(\frac{1}{4}|B-A|\right) \leqslant \frac{1}{16p}\frac{g'(\frac{1}{4}|B-A|)}{\frac{1}{4}|B-A|}|B-A|^2 \leqslant \frac{1}{16p}\frac{g'(|A+t[B-A]|)}{|A+t[B-A]|}|B-A|^2. \tag{44}$$

Since $p \geqslant 2$, we have

$$\frac{p-2}{16p}\frac{g'(|A+t[B-A]|)}{|A+t[B-A]|^3}\big(\big[A+t[B-A]\big]:[B-A]\big)^2 \geqslant 0,$$

and therefore we can add it to the right side of (44) and use (5)(ii) and (41) to obtain

$$g\left(\frac{1}{4}|B-A|\right) \leqslant \frac{1}{16p}\frac{\mathrm{d}^2}{\mathrm{d}t^2}g\big(\big|A+t[B-A]\big|\big) \tag{45}$$

when $p \geqslant 2$. Combining our estimates for the case $1<p<2$ and the case $p \geqslant 2$, we have established (40) for any $p>1$, every $x \in \mathcal{V}$, and a.e. $t \in [1/2, 3/4]$.

We now proceed with the original estimate. Using Lemma 3.1(vi) and (40), we obtain

$$\begin{aligned}\int_{\mathcal{V}} g'\big(|A|\big)|B-A|\,\mathrm{d}x &= 4\int_{\mathcal{V}} g'\big(|A|\big)\left(\frac{1}{4}|B-A|\right)\mathrm{d}x \\ &\leqslant C\int_{\mathcal{V}} g\big(|A|\big)\,\mathrm{d}x + C\int_{\mathcal{V}}\int_{1/2}^{3/4}(1-t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}g\big(\big|A+t[B-A]\big|\big)\,\mathrm{d}t\,\mathrm{d}x.\end{aligned}$$

Recalling that $\frac{\mathrm{d}^2}{\mathrm{d}t^2}g(|A+t[B-A]|) \geqslant 0$ for a.e. $t \in [0,1]$, since the function $t \mapsto g(|A+t[B-A]|)$ is convex, we can expand the domain of integration in the right side of the previous inequality to get

$$\begin{aligned}\int_{\mathcal{V}} g'\big(|A|\big)|B-A|\,\mathrm{d}x &\leqslant C\int_{\mathcal{V}} g\big(|A|\big)\,\mathrm{d}x + C\int_{\mathcal{V}}\int_{0}^{1}(1-t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}g\big(\big|A+t[B-A]\big|\big)\,\mathrm{d}t\,\mathrm{d}x \\ &= C\int_{\mathcal{V}} g\big(|A|\big)\,\mathrm{d}x + C\int_{\mathcal{V}}\left[g\big(|B|\big) - g\big(|A|\big) - \frac{\partial}{\partial F}g\big(|A|\big):[B-A]\right]\mathrm{d}x \\ &= C\int_{\mathcal{V}}\left\{g\big(|B|\big) - \frac{\partial}{\partial F}g\big(|A|\big):[B-A]\right\}\mathrm{d}x.\end{aligned}$$

Combining this estimate with the one in (38), the proof is complete. □

If f is asymptotically related to g, then we can use their similar asymptotic behavior to prove the following lemma, which states that if a function u is an almost minimizer for the functional with integrand f, then u is also an almost minimizer for the functional with integrand g.

Lemma 5.2. *Let $f:\mathbb{R}^{N\times n}\to\mathbb{R}$ be a function that is asymptotically related to g, where g is a function with (p,q)-structure, and suppose that f satisfies the growth condition*

$$\left|f(F)\right|\leqslant Lg\big(|F|\big)+\alpha$$

for some positive L and α. Let $A:\mathcal{U}\to\mathbb{R}^{N\times n}$ and $G\in\mathcal{C}^0(\overline{\mathcal{U}};\mathbb{R}^{n\times n})$ with matrix inverse $G^{-1}\in\mathcal{C}^0(\overline{\mathcal{U}};\mathbb{R}^{n\times n})$ be given, where $\mathcal{U}\subset\mathbb{R}^n$ is open and bounded. Define the functionals $J:W^{1,1}(\mathcal{U};\mathbb{R}^N)\to\mathbb{R}^*$ and $K:W^{1,1}(\mathcal{U};\mathbb{R}^N)\to\mathbb{R}^*$ by

$$J(w):=\int\limits_{\mathcal{U}} f\big([\nabla w+A]G\big)\,\mathrm{d}x\quad\textit{and}\quad K(w):=\int\limits_{\mathcal{U}} g\big(\big|[\nabla w+A]G\big|\big)\,\mathrm{d}x.$$

If u is a $(J,\{\omega_\varepsilon\},\{\nu_\varepsilon\})$-minimizer at x_0, then u is a $(K,\{\hat\omega_\varepsilon\},\{\hat\nu_\varepsilon\})$-minimizer at x_0, where $\hat\omega_\varepsilon$ and $\hat\nu_\varepsilon$ are defined as $\hat\omega_\varepsilon:=2\omega_{\varepsilon/4}$ and

$$\hat\nu_\varepsilon:=|\nu_{\varepsilon/4}|+(2L+1)g(\sigma_{\varepsilon/2})+2\alpha+\big(Lg(\sigma_1)+\alpha\big)\left(2\omega_\varepsilon\big(\operatorname{diam}(\mathcal{U})\big)+\frac{\varepsilon}{2}\right).$$

Proof. Since u is a $(J,\{\omega_\varepsilon\},\{\nu_\varepsilon\})$-minimizer at x_0, $J(u)<\infty$. Thus we can use Lemma 3.2 to get that $K(u)<\infty$. It remains to show (4). To this end, let $\varepsilon>0$ and $0<\rho<\operatorname{diam}(\mathcal{U})$, and fix $\varphi\in W_0^{1,1}(\mathcal{U}\cap\mathcal{B}_{x_0,\rho};\mathbb{R}^N)$. Upon writing $K(u)=J(u)+[K(u)-J(u)]$ and using the fact that u is a $(J,\{\omega_\varepsilon\},\{\nu_\varepsilon\})$-minimizer at x_0, we obtain

$$\begin{aligned}K(u)\leqslant{}& J(u+\varphi)+\left(\omega_{\frac{\varepsilon}{4}}(\rho)+\frac{\varepsilon}{4}\right)\int\limits_{\mathcal{U}\cap\mathcal{B}_{x_0,\rho}}\big\{\big|f\big([\nabla u+A]G\big)\big|+\big|f\big([\nabla u+\nabla\varphi+A]G\big)\big|\big\}\,\mathrm{d}x\\&+\int\limits_{\mathcal{U}\cap\mathcal{B}_{x_0,\rho}}\big\{\big|\nu_{\frac{\varepsilon}{4}}(x)\big|+\big|\nu_{\frac{\varepsilon}{4}}(x_0)\big|\big\}\,\mathrm{d}x+\int\limits_{\mathcal{U}}\big\{g\big(\big|[\nabla u+A]G\big|\big)-f\big([\nabla u+A]G\big)\big\}\,\mathrm{d}x.\end{aligned}\tag{46}$$

Note that

$$J(u+\varphi)=K(u+\varphi)+\int\limits_{\mathcal{U}}\big\{f\big([\nabla u+\nabla\varphi+A]G\big)-g\big(\big|[\nabla u+\nabla\varphi+A]G\big|\big)\big\}\,\mathrm{d}x.$$

Thus we can rewrite (46) as

$$\begin{aligned}K(u)\leqslant{}& K(u+\varphi)+\left(\omega_{\frac{\varepsilon}{4}}(\rho)+\frac{\varepsilon}{4}\right)\int\limits_{\mathcal{U}\cap\mathcal{B}_{x_0,\rho}}\big\{\big|f\big([\nabla u+A]G\big)\big|+\big|f\big([\nabla u+\nabla\varphi+A]G\big)\big|\big\}\,\mathrm{d}x\\&+\int\limits_{\mathcal{U}\cap\mathcal{B}_{x_0,\rho}}\big\{\big|\nu_{\varepsilon/4}(x)\big|+\big|\nu_{\varepsilon/4}(x_0)\big|\big\}\,\mathrm{d}x+\int\limits_{\mathcal{U}\cap\mathcal{B}_{x_0,\rho}}\big\{g\big(\big|[\nabla u+A]G\big|\big)-f\big([\nabla u+A]G\big)\big\}\,\mathrm{d}x\\&+\int\limits_{\mathcal{U}\cap\mathcal{B}_{x_0,\rho}}\big\{f\big([\nabla u+\nabla\varphi+A]G\big)-g\big(\big|[\nabla u+\nabla\varphi+A]G\big|\big)\big\}\,\mathrm{d}x\\={}&K(u+\varphi)+\left(\omega_{\varepsilon/4}(\rho)+\frac{\varepsilon}{4}\right)I_1+I_2+I_3+I_4,\end{aligned}\tag{47}$$

where we have noted that the last two integrals make opposite contributions outside $\mathcal{U} \cap \mathcal{B}_{x_0,\rho}$ and have defined $I_1, \dots, I_4$ in the obvious ways. Using the growth condition on f and the hypothesis that f is asymptotically related to g, we have the estimate $|f(F)| \leqslant Lg(\sigma_{\varepsilon^*}) + \alpha + (1+\varepsilon^*)g(|F|)$ for all $F \in \mathbb{R}^{N\times n}$ and $\varepsilon^* > 0$. Using this inequality in I_1 with $\varepsilon^* = 1$, we get

$$I_1 \leqslant 2 \int_{\mathcal{U}\cap\mathcal{B}_{x_0,\rho}} \{Lg(\sigma_1) + \alpha\}\,\mathrm{d}x + 2 \int_{\mathcal{U}\cap\mathcal{B}_{x_0,\rho}} \{g(|[\nabla u + A]G|) + g(|[\nabla u + \nabla\varphi + A]G|)\}\,\mathrm{d}x.$$

Since f is asymptotically related to g and f satisfies the growth condition, we get

$$I_3 \leqslant \int_{\mathcal{U}\cap\mathcal{B}_{x_0,\rho}} \{(L+1)g(\sigma_{\varepsilon/2}) + \alpha\}\,\mathrm{d}x + \frac{\varepsilon}{2} \int_{\mathcal{U}\cap\mathcal{B}_{x_0,\rho}} g(|[\nabla u + A]G|)\,\mathrm{d}x.$$

Proceeding similarly and noting that $-g(|\nabla u + \nabla\varphi + A|G) \leqslant 0$, we obtain

$$I_4 \leqslant \int_{\mathcal{U}\cap\mathcal{B}_{x_0,\rho}} \{Lg(\sigma_{\varepsilon/2}) + \alpha\}\,\mathrm{d}x + \frac{\varepsilon}{2} \int_{\mathcal{U}\cap\mathcal{B}_{x_0,\rho}} g(|[\nabla u + \nabla\varphi + A]G|)\,\mathrm{d}x.$$

Putting our estimates for I_1, I_3, and I_4 into (47) and defining $\hat{\omega}_\varepsilon$ and $\hat{\nu}_\varepsilon$ as in the statement of the lemma, we obtain

$$\begin{aligned} K(u) \leqslant K(u+\varphi) &+ (\hat{\omega}_\varepsilon(\rho) + \varepsilon) \int_{\mathcal{U}\cap\mathcal{B}_{x_0,\rho}} \{g(|[\nabla u + A]G|) + g(|[\nabla u + \nabla\varphi + A]G|)\}\,\mathrm{d}x \\ &+ \int_{\mathcal{U}\cap\mathcal{B}_{x_0,\rho}} \{\hat{\nu}_\varepsilon(x) + \hat{\nu}_\varepsilon(y)\}\,\mathrm{d}x, \end{aligned}$$

and hence u is a $(K, \{\hat{\omega}_\varepsilon\}, \{\hat{\nu}_\varepsilon\})$-minimizer at x_0. $\square$

The following lemma is the crux of the argument that establishes Sobolev–Morrey regularity for almost minimizers of (1).

Lemma 5.3. *Suppose $\{g_y\}_{y\in\mathcal{B}^+}$ is a family of functions with uniform (p,q)-structure, where $1 < p \leqslant q$, and let $g : [0,\infty) \to [0,\infty)$ be as in Definition 2.7. Let $A \in L^{g,\kappa}(\mathcal{B}^+; \mathbb{R}^{N\times n})$ and $G \in \mathcal{C}^0(\overline{\mathcal{B}^+}; \mathbb{R}^{n\times n})$ with matrix inverse $G^{-1} \in \mathcal{C}^0(\overline{\mathcal{B}^+}; \mathbb{R}^{n\times n})$ be given. For every $y \in \mathcal{B}^+$, define $K_y^+ : W^{1,1}(\mathcal{B}^+; \mathbb{R}^N) \to \mathbb{R}^*$ by*

$$K_y^+(w) := \int_{\mathcal{B}^+} g_y(|[\nabla w + A]G|)\,\mathrm{d}x.$$

Suppose that $u \in W^{1,1}(\mathcal{B}^+; \mathbb{R}^N)$ is a $(K_y, \{\omega_\varepsilon\}, \{\nu_\varepsilon\})$-minimizer at y for each $y \in \mathcal{B}^+$, where $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\kappa}(\mathcal{B}^+)$, and that $u = 0$ on $\mathcal{B} \cap \partial H^+$ in the sense of trace. Then $u \in W^{1,(p,\kappa)}_{\mathrm{loc}}(\mathcal{B}^{\mp}; \mathbb{R}^N)$ and $\nabla u \in L^{g,\kappa}_{\mathrm{loc}}(\mathcal{B}^{\mp}; \mathbb{R}^{N\times n})$.

Proof. To begin, we define a few items for notational convenience. We will let G_y denote the matrix $G(y)$, and $R_y := 1 - |y|$. Let

$$M := \sup_{x\in\overline{\mathcal{B}^+}} \{|G(x)|\} + \sup_{x\in\overline{\mathcal{B}^+}} \{|G^{-1}(x)|\} + 1,$$

and define $\mu : [0,1] \to [0,\infty)$ by

$$\mu(s) := \sup_{x,z\in\mathcal{B}^+} \big\{ \big|G(x)-G(z)\big|:\ |x-z| \leqslant s \big\}.$$

Note that $\mu \in \mathcal{C}^0([0,1])$ is nondecreasing and satisfies $\mu(0)=0$. Throughout this proof, we will denote by C a generic constant that can depend on n, m, p, q, and M. With these notations in place, we are now set up to begin the proof. The proof contains two main steps. We will first show that there is a constant c_4, independent of x_0, such that

$$\int_{\mathcal{B}_{x_0,\rho}} g\big(|\nabla u G_{x_0}|\big)\,\mathrm{d}x \leqslant c_4\left(\frac{\rho}{R}\right)^{\kappa} \int_{\mathcal{B}_{x_0,R}} g\big(|\nabla u G_{x_0}|\big)\,\mathrm{d}x + c_4\rho^{\kappa} \tag{48}$$

for every $x_0 \in \mathcal{B}^+$ and $0<\rho \leqslant R \leqslant R_{x_0}$. We then use this estimate to demonstrate that $u \in W^{1,(p,\kappa)}_{\mathrm{loc}}(\mathcal{B}^{\mp};\mathbb{R}^N)$ and $\nabla u \in L^{g,\kappa}_{\mathrm{loc}}(B^{\mp};\mathbb{R}^{N\times n})$.

For fixed $x_0 \in \mathcal{B}^+$, suppose that $0<\rho \leqslant R/2 < R \leqslant \frac{1}{2}R_{x_0}$, and let $y \in \mathcal{B}^+_{x_0,\frac{1}{2}R_{x_0}}$. Define $I: W^{1,1}(\mathcal{B}^+_{y,R},\mathbb{R}^N) \to \mathbb{R}^*$ by $I(w) := \int_{\mathcal{B}^+_{y,R}} g_y(|\nabla w G_y|)\,\mathrm{d}x$. Let v be a minimizer for I satisfying $u-v \in W^{1,p}_0(\mathcal{B}^+_{y,R};\mathbb{R}^N)$. We have

$$\begin{aligned}\int_{\mathcal{B}^+_{y,\rho}} g_y\big(|\nabla u G_y|\big)\,\mathrm{d}x &= \int_{\mathcal{B}^+_{y,\rho}} g_y\big(|\nabla v G_y|\big)\,\mathrm{d}x + \int_{\mathcal{B}^+_{y,\rho}} \big\{g_y\big(|\nabla u G_y|\big) - g_y\big(|\nabla v G_y|\big)\big\}\,\mathrm{d}x \\ &\leqslant C\rho^n \big\|g_y\big(|\nabla v G_y|\big)\big\|_{L^\infty(\mathcal{B}^+_{y,\rho})} + \int_{\mathcal{B}^+_{y,\rho}} \big\{g_y\big(|\nabla u G_y|\big) - g_y\big(|\nabla v G_y|\big)\big\}\,\mathrm{d}x. \end{aligned} \tag{49}$$

Recalling that $\rho \leqslant R/2$, by Theorem 4.3, we have that

$$\big\|g_y\big(|\nabla v G_y|\big)\big\|_{L^\infty(\mathcal{B}^+_{y,\rho})} \leqslant \frac{C}{R^n} \int_{\mathcal{B}^+_{y,R}} g_y\big(|\nabla v G_y|\big).$$

Using the above inequality and the fact that $I(v) \leqslant I(u)$, we get that

$$\begin{aligned} C\rho^n \big\|g_y\big(|\nabla v G_y|\big)\big\|_{L^\infty(\mathcal{B}^+_{y,\rho})} &\leqslant C\left(\frac{\rho}{R}\right)^n \int_{\mathcal{B}^+_{y,R}} g_y\big(|\nabla v G_y|\big)\,\mathrm{d}x \\ &\leqslant C\left(\frac{\rho}{R}\right)^n \int_{\mathcal{B}^+_{y,R}} g_y\big(|\nabla u G_y|\big)\,\mathrm{d}x. \end{aligned} \tag{50}$$

Combining (49) and (50) yields

$$\begin{aligned}\int_{\mathcal{B}^+_{y,\rho}} g_y\big(|\nabla u G_y|\big)\,\mathrm{d}x &\leqslant C\left(\frac{\rho}{R}\right)^n \int_{\mathcal{B}^+_{y,R}} g_y\big(|\nabla u G_y|\big)\,\mathrm{d}x \\ &\quad + \int_{\mathcal{B}^+_{y,\rho}} \big\{g_y\big(|\nabla u G_y|\big) - g_y\big(|\nabla v G_y|\big)\big\}\,\mathrm{d}x. \end{aligned} \tag{51}$$

We will now estimate the last integral. We can write

$$\int_{\mathcal{B}^+_{y,\rho}} \big\{g_y\big(|\nabla u G_y|\big) - g_y\big(|\nabla v G_y|\big)\big\}\,\mathrm{d}x = I_1 + I_2,$$

where we have defined

$$I_1 := \int_{\mathcal{B}^+_{y,\rho}} \left\{g_y\big(\big|\nabla u G_y\big|\big) - g_y\big(|\nabla v G_y|\big) - \frac{\partial}{\partial F} g_y\big(|\nabla v G_y|\big) : [\nabla u - \nabla v]G_y\right\}\mathrm{d}x;$$

$$I_2 := \int_{\mathcal{B}^+_{y,\rho}} \frac{\partial}{\partial F} g_y\big(|\nabla v G_y|\big) : [\nabla u - \nabla v]G_y\,\mathrm{d}x.$$

By the convexity of g_y, the integrand in I_1 is non-negative, so we can expand the domain of integration to $\mathcal{B}^+_{y,R}$ and then apply Lemma 3.5 to get

$$I_1 \leqslant \int_{\mathcal{B}^+_{y,R}} \big[g_y\big(|\nabla u G_y|\big) - g_y\big(|\nabla v G_y|\big)\big]\,\mathrm{d}x. \tag{52}$$

Now we estimate I_2. Since $|\frac{\partial}{\partial F} g_y(|F|)| = g'_y(|F|)$, we find that

$$I_2 \leqslant \int_{\mathcal{B}^+_{y,\rho}} g'_y\big(|\nabla v G_y|\big)\big|[\nabla u - \nabla v]G_y\big|\,\mathrm{d}x.$$

From the above inequality and Lemma 5.1, we obtain

$$\begin{aligned} I_2 &\leqslant C \int_{\mathcal{B}^+_{y,\rho}} \left\{g_y\big(|\nabla u G_y|\big) - \frac{\partial}{\partial F} g_y\big(|\nabla v G_y|\big) : [\nabla u - \nabla v]G_y\right\}\mathrm{d}x \\ &= C\left[\int_{\mathcal{B}_{y,\rho}} g_y\big(|\nabla v G_y|\big)\,\mathrm{d}x + I_1\right] \leqslant C\big[\rho^n \big\|g_y\big(|\nabla v G_y|\big)\big\|_{L^\infty(\mathcal{B}_{y,\rho})} + I_1\big], \end{aligned}$$

where I_1 is as defined above. Using (50) and (52), we deduce that

$$I_2 \leqslant C\left(\frac{\rho}{R}\right)^n \int_{\mathcal{B}^+_{y,R}} g_y\big(|\nabla u G_y|\big)\,\mathrm{d}x + C\int_{\mathcal{B}^+_{y,R}} \big\{g_y\big(|\nabla u G_y|\big) - g\big(|\nabla v G_y|\big)\big\}\,\mathrm{d}x.$$

Collecting our estimates for I_1 and I_2, we have so far proven that

$$\begin{aligned} \int_{\mathcal{B}^+_{y,\rho}} g_y\big(|\nabla u G_y|\big)\,\mathrm{d}x &\leqslant C\left(\frac{\rho}{R}\right)^n \int_{\mathcal{B}^+_{y,R}} g_y\big(|\nabla u G_y|\big)\,\mathrm{d}x \\ &\quad + C\int_{\mathcal{B}^+_{y,R}} \big\{g_y\big(|\nabla u G_y|\big) - g_y\big(|\nabla v G_y|\big)\big\}\,\mathrm{d}x. \end{aligned} \tag{53}$$

But u is a $(K_y, \{\omega_\varepsilon\}, \{\nu_\varepsilon\})$-minimizer at y and $\nu_\varepsilon \in L^{1,\kappa}(\mathcal{B}^+)$, which implies

$$\begin{aligned}
&\int\limits_{\mathcal{B}^+_{y,R}} \big\{g_y\big(|\nabla u G_y|\big) - g_y\big(|\nabla v G_y|\big)\big\}\,\mathrm{d}x \\
&\quad\leqslant \int\limits_{\mathcal{B}^+_{y,R}} \big\{g_y\big(|\nabla u G_y|\big) - g_y\big(\big|[\nabla u + A]G\big|\big)\big\}\,\mathrm{d}x + \int\limits_{\mathcal{B}^+_{y,R}} \big\{g_y\big(\big|[\nabla v + A]G\big|\big) - g_y\big(|\nabla v G_y|\big)\big\}\,\mathrm{d}x \\
&\qquad + \big(\omega_\varepsilon(R) + \varepsilon\big) \int\limits_{\mathcal{B}^+_{y,R}} \big\{g_y\big(\big|[\nabla u + A]G\big|\big) + g_y\big(\big|[\nabla v + A]G\big|\big)\big\}\,\mathrm{d}x + R^\kappa \|\nu_\varepsilon\|_{L^{1,\kappa}} + R^n\big|\nu_\varepsilon(y)\big| \\
&\quad = I_3 + I_4 + \big(\omega_\varepsilon(R) + \varepsilon\big) I_5 + R^\kappa \|\nu_\varepsilon\|_{L^{1,\kappa}} + R^n\big|\nu_\varepsilon(y)\big|,
\end{aligned} \tag{54}$$

where I_3, I_4, and I_5 are defined in the obvious ways. We first estimate I_3. Note that we have

$$\begin{aligned}
I_3 &= \int\limits_{\mathcal{B}^+_{y,R}} \int\limits_0^1 \frac{\partial}{\partial F} g_y\big(\big|\nabla u G + t\nabla u(G_y - G) + (1-t)AG\big|\big) : \big[\nabla u(G_y - G) - AG\big]\,\mathrm{d}t\,\mathrm{d}x \\
&\leqslant \int\limits_{\mathcal{B}^+_{y,R}} \int\limits_0^1 g_y'\big(C|\nabla u G_y| + C|A|\big)\big[C|\nabla u G_y|\mu(R) + C|A|\big]\,\mathrm{d}t\,\mathrm{d}x \\
&\leqslant C\varepsilon \int\limits_{\mathcal{B}^+_{y,R}} g_y'\big(C|\nabla u G_y| + C|A|\big)\left[\frac{1}{\varepsilon}\big(|\nabla u G_y|\mu(R) + |A|\big)\right]\mathrm{d}x
\end{aligned}$$

for any $0 < \varepsilon < 1$. Using Lemma 3.1(vi), we obtain

$$\begin{aligned}
&C\varepsilon \int\limits_{\mathcal{B}^+_{y,R}} g_y'\big(C|\nabla u G_y| + C|A|\big)\left[\frac{1}{\varepsilon}\big(|\nabla u G_y|\mu(R) + |A|\big)\right]\mathrm{d}x \\
&\quad\leqslant C\varepsilon \int\limits_{\mathcal{B}^+_{y,R}} g_y\big(C|\nabla u G_y| + C|A|\big)\,\mathrm{d}x + C\varepsilon \int\limits_{\mathcal{B}^+_{y,R}} g_y\left(\frac{1}{\varepsilon}\big(|\nabla u G_y|\mu(R) + |A|\big)\right)\mathrm{d}x.
\end{aligned}$$

Since $\{g_y\}_{y\in\mathcal{B}^+}$ has uniform (p,q)-structure, there is a constant c such that

$$g(s) - c \leqslant g_y(s) \leqslant c g(s) + c \tag{55}$$

for all $s \geqslant 0$. Using (55) along with parts (iii) and (v) of Lemma 3.1, we finally get

$$I_3 \leqslant C\left(\varepsilon + \frac{\mu(R)}{\varepsilon^{q-1}}\right) \int\limits_{\mathcal{B}^+_{y,R}} g_y\big(|\nabla u G_y|\big)\,\mathrm{d}x + \frac{CcR^\kappa}{\varepsilon^{q-1}}\big\|g\big(|A|\big)\big\|_{L^{1,\kappa}} + Cc\varepsilon R^n.$$

Using a similar computation and the fact that v is a minimizer for I, we get

$$I_4 \leqslant C\left(\varepsilon + \frac{\mu(R)}{\varepsilon^{q-1}}\right) \int\limits_{\mathcal{B}^+_{y,R}} g_y\big(|\nabla u G_y|\big)\,\mathrm{d}x + \frac{CcR^\kappa}{\varepsilon^{q-1}} \big\|g\big(|A|\big)\big\|_{L^{1,\kappa}} + Cc\varepsilon R^n.$$

To estimate I_5, we use (55), parts (iii) and (v) of Lemma 3.1, and the fact that v minimizes I to obtain

$$I_5 \leqslant C \int\limits_{\mathcal{B}^+_{y,R}} g_y\big(|\nabla u G_y|\big)\,\mathrm{d}x + CcR^\kappa \big\|g\big(|A|\big)\big\|_{L^{1,\kappa}} + CcR^n.$$

Upon substituting the estimates for I_3, I_4, and I_5 into (54), we have

$$\begin{aligned} &\int\limits_{\mathcal{B}^+_{y,R}} \big\{g_y\big(|\nabla u G_y|\big) - g_y\big(|\nabla v G_y|\big)\big\}\,\mathrm{d}x \\ &\quad\leqslant C\big(\varepsilon + \omega_\varepsilon(R) + \mu_\varepsilon(R)\big) \int\limits_{\mathcal{B}^+_{y,R}} g_y\big(|\nabla u G_y|\big)\,\mathrm{d}x + c_{1,\varepsilon} R^\kappa + C\big|\nu_\varepsilon(y)\big|R^n, \end{aligned} \tag{56}$$

where we have put $\mu_\varepsilon := \mu/\varepsilon^{q-1}$ and where $c_{1,\varepsilon}$ is a constant that depends on n, p, q, c, ε, and $\omega_\varepsilon(1)$. Using (53)–(56), we deduce that

$$\int\limits_{\mathcal{B}^+_{y,\rho}} g\big(|\nabla u G_y|\big)\,\mathrm{d}x \leqslant Cc\left(\left(\frac{\rho}{R}\right)^n + \varepsilon + \omega_\varepsilon(R) + \mu_\varepsilon(R)\right) \int\limits_{\mathcal{B}^+_{y,R}} g\big(|\nabla u G_y|\big)\,\mathrm{d}x + c_{2,\varepsilon} R^\kappa + C\big|\nu_\varepsilon(y)\big|R^n, \tag{57}$$

where $c_{2,\varepsilon}$ depends on n, p, q, M, ε, c, $\omega_\varepsilon(1)$, $\mu_\varepsilon(1)$, $\|g(|A|)\|_{L^{1,\kappa}}$, and $\|\nu_\varepsilon\|_{L^{1,\kappa}}$.

This inequality is valid for every y in $\mathcal{B}^+_{x_0,\frac{1}{2}R_{x_0}}$, so we can integrate (57) over $\mathcal{B}^+_{x_0,\rho}$ with respect to the center y, which results in

$$\begin{aligned} \int\limits_{\mathcal{B}^+_{x_0,\rho}} \int\limits_{\mathcal{B}^+_{y,\rho}} g\big(|\nabla u G_y|\big)\,\mathrm{d}x\,\mathrm{d}y &\leqslant Cc\left(\left(\frac{\rho}{R}\right)^n + \varepsilon + \omega_\varepsilon(R) + \mu_\varepsilon(R)\right) \int\limits_{\mathcal{B}^+_{x_0,\rho}} \int\limits_{\mathcal{B}^+_{y,R}} g\big(|\nabla u G_y|\big)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int\limits_{\mathcal{B}^+_{x_0,\rho}} \big\{c_{2,\varepsilon} R^\kappa + \big|\nu_\varepsilon(y)\big|R^n\big\}\,\mathrm{d}y. \end{aligned} \tag{58}$$

We estimate the double integral on the left from below using Lemma 3.1(iii):

$$\begin{aligned} \int\limits_{\mathcal{B}^+_{x_0,\rho}} \int\limits_{\mathcal{B}^+_{y,\rho}} g\big(|\nabla u G_y|\big)\,\mathrm{d}x\,\mathrm{d}y &\geqslant \frac{1}{C} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} g\big(|\nabla u G_{x_0}|\big) \chi_{\mathcal{B}^+_{y,\rho}}(x) \chi_{\mathcal{B}^+_{x_0,\frac{\rho}{2}}}(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\geqslant \frac{1}{C} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} g\big(|\nabla u G_{x_0}|\big) \chi_{\mathcal{B}^+_{x_0,\frac{\rho}{2}}}(x) \chi_{\mathcal{B}^+_{x_0,\frac{\rho}{2}}}(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\geqslant \frac{\rho^n}{C} \int\limits_{\mathcal{B}^+_{x_0,\frac{\rho}{2}}} g\big(|\nabla u G_{x_0}|\big)\,\mathrm{d}x. \end{aligned} \tag{59}$$

We estimate the double integral on the right from above similarly:

$$\int\limits_{\mathcal{B}^+_{x_0,\rho}}\int\limits_{\mathcal{B}^+_{y,R}} g\big(|\nabla u G_y|\big)\,\mathrm{d}x\,\mathrm{d}y \leqslant C\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} g\big(|\nabla u G_{x_0}|\big)\chi_{\mathcal{B}^+_{y,R}}(x)\chi_{\mathcal{B}^+_{x_0,\rho}}(y)\,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant C\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} g\big(|\nabla u G_{x_0}|\big)\chi_{\mathcal{B}^+_{x_0,R+\rho}}(x)\chi_{\mathcal{B}^+_{x_0,\rho}}(y)\,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant C\rho^n \int\limits_{\mathcal{B}^+_{x_0,2R}} g\big(|\nabla u G_{x_0}|\big)\,\mathrm{d}x. \tag{60}$$

For the last integral in (58), we have

$$\int\limits_{\mathcal{B}^+_{x_0,\rho}} \big\{c_{2,\varepsilon}R^\kappa + \big|\nu_\varepsilon(y)\big|R^n\big\}\,\mathrm{d}y \leqslant Cc_{2,\varepsilon}R^{n+\kappa} + \|\nu_\varepsilon\|_{L^{1,\kappa}}R^{n+\kappa} =: c_{3,\varepsilon}R^{n+\kappa}. \tag{61}$$

Putting our estimates from (59)–(61) into (58) and defining $c_4 := Cc$, we get

$$\int\limits_{\mathcal{B}^+_{x_0,\frac{\rho}{2}}} g\big(|\nabla u G_{x_0}|\big)\,\mathrm{d}x \leqslant c_4\left(\left(\frac{\rho}{R}\right)^n + \varepsilon + \omega_\varepsilon(R) + \frac{\mu(R)}{\varepsilon^{2q-1}}\right)\int\limits_{\mathcal{B}_{x_0,2R}} g\big(|\nabla u G_{x_0}|\big)\,\mathrm{d}x + c_{3,\varepsilon}\frac{R^{n+\kappa}}{\rho^n}. \tag{62}$$

We have shown that (62) holds for all $0<\rho\leqslant R/2<R\leqslant R_{x_0}/2$, and hence

$$\int\limits_{\mathcal{B}^+_{x_0,\rho}} g\big(|\nabla u G_{x_0}|\big)\,\mathrm{d}x \leqslant c_4\left(\left(\frac{\rho}{R}\right)^n + \varepsilon + \omega_\varepsilon(R) + \mu_\varepsilon(R)\right)\int\limits_{\mathcal{B}^+_{x_0,R}} g\big(|\nabla u G_{x_0}|\big)\,\mathrm{d}x + c_{3,\varepsilon}\frac{R^{n+\kappa}}{\rho^n} \tag{63}$$

for all $0<\rho\leqslant R/8<R\leqslant R_{x_0}$. But by enlarging c_4 if necessary, we can see that (63) obviously holds for all $0<R/8<\rho\leqslant R\leqslant R_{x_0}$; therefore we have that (63) holds for all $0<\rho\leqslant R\leqslant R_{x_0}$.

Now we will use Lemma 3.4 to obtain (48). Fix $\varepsilon^* := 1/(2c_4)^{\frac{2n}{n-\kappa}}$, and define the function $\tilde{R}:\mathcal{B}^{\mp}\to(0,1]$ by

$$\tilde{R}(x) = \sup\left\{R\in\big(0,1-|x|\big]\colon\ \omega_{\varepsilon^*/2}(R)+\mu_{\varepsilon^*/2}(R)\leqslant\frac{\varepsilon^*}{2}\right\}.$$

Thus, defining $c_5 := \max\{c_4, c_{3,\varepsilon^*/2}\}$, we have

$$\int\limits_{\mathcal{B}^+_{x_0,\rho}} g\big(|\nabla u G_{x_0}|\big)\,\mathrm{d}x \leqslant c_5\left[\left(\frac{\rho}{R}\right)^n + \varepsilon^*\right]\int\limits_{\mathcal{B}^+_{x_0,R}} g\big(|\nabla u G_{x_0}|\big)\,\mathrm{d}x + c_5\frac{R^{n+\kappa}}{\rho^n}$$

for all $0<\rho\leqslant R\leqslant\tilde{R}(x_0)$. According to Lemma 3.4, we may now write

$$\int\limits_{\mathcal{B}^+_{x_0,\rho}} g\big(|\nabla u G_{x_0}|\big)\,\mathrm{d}x \leqslant c_6\left(\frac{\rho}{R}\right)^\kappa\int\limits_{\mathcal{B}^+_{x_0,R}} g\big(|\nabla u G_{x_0}|\big)\,\mathrm{d}x + c_6\rho^\kappa$$

for all $0 < \rho \leqslant R \leqslant \tilde{R}(x_0)$ and some finite constant c_6. But by the definition of $\tilde{R}$, we see that if we define $r = \sup_{x \in \mathcal{B}^{\mp}}\{\tilde{R}(x)\}$, then $\tilde{R}(x_0) = \min\{r, 1 - |x_0|\}$. Putting $c_7 := c_6/r^{\kappa}$, we have

$$\int_{\mathcal{B}^+_{x_0,\rho}} g\big(|\nabla u G_{x_0}|\big)\,\mathrm{d}x \leqslant c_7\left(\frac{\rho}{R}\right)^{\kappa} \int_{\mathcal{B}^+_{x_0,R}} g\big(|\nabla u G_{x_0}|\big)\,\mathrm{d}x + c_7\rho^{\kappa}$$

for all $0 < \rho \leqslant R \leqslant 1 - |x_0|$. This establishes (48).

Now we use (48) to show $u \in W^{1,(p,\kappa)}_{\mathrm{loc}}(\mathcal{B}^{\mp}; \mathbb{R}^N)$ and $\nabla u \in L^{g,\kappa}_{\mathrm{loc}}(\mathcal{B}^{\mp}; \mathbb{R}^{N\times n})$. Since g has (p,q)-structure, we can employ Lemma 3.1(iii) to estimate the left side of (48) from below:

$$\int_{\mathcal{B}^+_{x_0,\rho}} g\big(|\nabla u G_{x_0}|\big)\,\mathrm{d}x \geqslant \frac{1}{C} \int_{\mathcal{B}^+_{x_0,\rho}} g\big(|\nabla u|\big)\,\mathrm{d}x. \tag{64}$$

We also estimate the integral on the right from above. Fix $y_0 \in \mathcal{B}^+$, and note that $g(t) \leqslant g_{y_0}(t) + g(1)$ for all $t \geqslant 0$. Using this and Lemma 3.1, we have

$$\begin{aligned}\int_{\mathcal{B}^+_{x_0,R}} g\big(|\nabla u G_{x_0}|\big)\,\mathrm{d}x &\leqslant \int_{\mathcal{B}^+} g_{y_0}\big(|\nabla u G_{x_0}|\big)\,\mathrm{d}x + Cg(1)\\ &\leqslant C\int_{\mathcal{B}^+} g_{y_0}\big(\big|[\nabla u + A]G\big|\big)\,\mathrm{d}x + C\big\|g_{y_0}\big(|A|\big)\big\|_{L^1} + Cg(1) =: c_8.\end{aligned} \tag{65}$$

Note that $c_8 < \infty$, since $\int_{\mathcal{B}^+} g_{y_0}(|[\nabla u + A]G|)\,\mathrm{d}x = K_{y_0}(u) < \infty$. Collecting our estimates in (64) and (65) into (48), and dividing both sides by ρ^{κ}, we finally arrive at

$$\frac{1}{\rho^{\kappa}} \int_{\mathcal{B}^+_{x_0,\rho}} g\big(|\nabla u|\big)\,\mathrm{d}x \leqslant \frac{Cc_7c_8}{(1-|x_0|)^{\kappa}} + Cc_7$$

for all $0 < \rho \leqslant 1 - |x_0|$, where we have taken $R = 1 - |x_0|$. But for any $\mathcal{U} \Subset \mathcal{B}^{\mp}$, we have that $1 - |x_0|$ is bounded away from 0 for all $x_0 \in \mathcal{U}$. Hence we have that $\nabla u \in L^{g,\kappa}_{\mathrm{loc}}(\mathcal{B}^{\mp}; \mathbb{R}^{N\times n})$, which implies that $\nabla u \in L^{p,\kappa}_{\mathrm{loc}}(\mathcal{B}^{\mp}; \mathbb{R}^{N\times n})$. Since $u \in W^{1,1}(\mathcal{B}^+; \mathbb{R}^N)$ and $u = 0$ on $\partial\mathcal{H}^+$, we can extend u via a negative reflection across $\partial\mathcal{H}^+$ and apply the Sobolev–Poincaré inequality to get $u \in L^{p,\kappa}_{\mathrm{loc}}(\mathcal{B}^{\mp}; \mathbb{R}^N)$. Therefore $u \in W^{1,(p,\kappa)}_{\mathrm{loc}}(\mathcal{B}^{\mp}; \mathbb{R}^N)$, and the proof is complete. □

Using Lemmas 5.2 and 5.3, we may state the following lemma.

Lemma 5.4. *Suppose the family of functions $\{f_y\}_{y \in \mathcal{B}^+}$, defined on $\mathbb{R}^{N\times n}$, is $L^{g,\kappa}$-asymptotically related to a family $\{g_y\}_{y\in\mathcal{B}^+}$ with uniform (p,q)-structure, and let g be as in Definition 2.7. Suppose also that there is an $L \geqslant 1$ and a function $\alpha \in L^{1,\kappa}(\mathcal{B}^+)$ such that*

$$f_y(F) \leqslant Lg\big(|F|\big) + \alpha(y)$$

for each $y \in \mathcal{B}^+$. Let $A \in L^{g,\kappa}(\mathcal{B}^+; \mathbb{R}^{N\times n})$ and $G \in C^0(\overline{\mathcal{B}^+}; \mathbb{R}^{n\times n})$ with matrix inverse $G^{-1} \in C^0(\overline{\mathcal{B}^+}; \mathbb{R}^{n\times n})$ be given. For each $y \in \mathcal{B}^+$, define $J^+_y : W^{1,1}(\mathcal{B}^+; \mathbb{R}^N) \to \mathbb{R}^*$ by

$$J^+_y(w) := \int_{\mathcal{B}^+} f_y\big([\nabla w + A]G\big)\,\mathrm{d}x.$$

Let $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\kappa}(\mathcal{B}^+)$ be given. Suppose that $u \in W^{1,1}(\mathcal{B}^+;\mathbb{R}^N)$ satisfies $u=0$ on $\mathcal{B} \cap \partial H^+$ in the sense of trace, and that u is a $(J_y^+, \{\omega_\varepsilon\}, \{\nu_\varepsilon\})$-minimizer at y for each $y \in \mathcal{B}^+$. Then $u \in W^{1,(p,\kappa)}_{\mathrm{loc}}(\mathcal{B}^{\mp};\mathbb{R}^N)$ and $\nabla u \in L^{g,\kappa}_{\mathrm{loc}}(\mathcal{B}^{\mp};\mathbb{R}^{N\times n})$.

Using Theorem 4.2 instead of Theorem 4.3, the following theorem can be established in the same way as Lemma 5.4.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Suppose that the family of functions $\{f_y\}_{y\in\Omega}$, defined on $\mathbb{R}^{N\times n}$, is $L^{g,\kappa}_{\mathrm{loc}}$-asymptotically related to a family $\{g_y\}_{y\in\Omega}$ with uniform (p,q)-structure, and let g be as in Definition 2.7. Suppose also that there is an $L \geqslant 1$ and a function $\alpha \in L^{1,\kappa}_{\mathrm{loc}}(\Omega)$ such that*

$$f_y(F) \leqslant Lg\big(|F|\big) + \alpha(y)$$

for each $y \in \Omega$. Let the mappings $A \in L^{g,\kappa}_{\mathrm{loc}}(\Omega;\mathbb{R}^{N\times n})$ and $G \in \mathcal{C}^0(\Omega;\mathbb{R}^{n\times n})$, with matrix inverse $G^{-1} \in \mathcal{C}^0(\Omega;\mathbb{R}^{n\times n})$, be given. For each $y \in \Omega$, define the functional $K_y : W^{1,1}(\Omega;\mathbb{R}^N) \to \mathbb{R}^*$ by

$$K_y[w] := \int\limits_\Omega f_y\big([\nabla w + A]G\big)\,\mathrm{d}x.$$

Let $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\kappa}_{\mathrm{loc}}(\Omega)$ be given, and suppose that $u \in W^{1,1}(\Omega;\mathbb{R}^N)$ is a local $(K_y, \{\omega_\varepsilon\}, \{\nu_\varepsilon\})$-minimizer at y for each $y \in \Omega$. Then $u \in W^{1,(p,\kappa)}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and $\nabla u \in L^{g,\kappa}_{\mathrm{loc}}(\Omega;\mathbb{R}^{N\times n})$.

A standard argument may be used to "straighten out" smooth portions of the boundary $\partial\Omega$. Thus Lemma 5.4 and Theorem 5.1 can be used to prove the following result.

Theorem 5.2. *Suppose that $\Omega \subset \mathbb{R}^n$ is an open and bounded set and that Γ is a $\mathcal{C}^1$ portion of $\partial\Omega$. Suppose also that the family of functions $\{f_y\}_{y\in\Omega}$, defined on $\mathbb{R}^{N\times n}$, is $L^{g,\kappa}_{\mathrm{loc}}(\Omega \cup \Gamma)$-asymptotically related to a family $\{g_y\}_{y\in\Omega}$ with uniform (p,q)-structure, where g is as in Definition 2.7. Suppose that there is an $L \geqslant 1$ and a function $\alpha \in L^{1,\kappa}_{\mathrm{loc}}(\Omega \cup \Gamma)$ such that*

$$f_y(F) \leqslant Lg\big(|F|\big) + \alpha(y)$$

for each $y \in \Omega$. Let the mappings $A \in L^{g,\kappa}_{\mathrm{loc}}(\Omega \cup \Gamma;\mathbb{R}^{N\times n})$ and $G \in \mathcal{C}^0(\Omega \cup \Gamma;\mathbb{R}^{n\times n})$, with matrix inverse $G^{-1} \in \mathcal{C}^0(\Omega \cup \Gamma;\mathbb{R}^{n\times n})$, be given. For each $y \in \Omega$, define the functional $K_y : W^{1,1}(\Omega;\mathbb{R}^N) \to \mathbb{R}^*$ by

$$K_y[w] := \int\limits_\Omega f_y\big([\nabla w + A]G\big)\,\mathrm{d}x.$$

Let $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\kappa}_{\mathrm{loc}}(\Omega \cup \Gamma)$ be given. If $u \in W^{1,1}(\Omega;\mathbb{R}^N)$ satisfies $u=0$ on Γ and is a $(K_y, \{\omega_\varepsilon\}, \{\nu_\varepsilon\})$-minimizer at y for each $y \in \Omega$, then $u \in W^{1,(p,\kappa)}_{\mathrm{loc}}(\Omega \cup \Gamma;\mathbb{R}^N)$ and $\nabla u \in L^{g,\kappa}_{\mathrm{loc}}(\Omega \cup \Gamma;\mathbb{R}^{N\times n})$.

6. Nonhomogeneous functionals

We now use the results from the previous section to show regularity for almost minimizers of functionals of the form $u \mapsto \int_\Omega f(x,\nabla u)\,\mathrm{d}x$.

Theorem 6.1. *Suppose that $\Omega \subset \mathbb{R}^n$ is an open, bounded set and that Γ is a $\mathcal{C}^1$ portion of $\partial\Omega$. Let $\{\nu_\varepsilon, \zeta_\varepsilon\}_{\varepsilon>0} \subset L^{1,\kappa}_{\mathrm{loc}}(\Omega \cup \Gamma)$ and $\bar{u} \in W^{1,1}_{\mathrm{loc}}(\Omega \cup \Gamma;\mathbb{R}^N)$ satisfying $\nabla\bar{u} \in L^{g,\kappa}_{\mathrm{loc}}(\Omega \cup \Gamma;\mathbb{R}^N)$ be given. Suppose* *that $\delta \in \mathcal{C}^0([0,\infty))$ and $\{\omega_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}^0([0,\infty))$ are nondecreasing functions satisfying $\delta(0) = \omega_\varepsilon(0) = 0$ for each $\varepsilon > 0$. Let $\{g_x\}_{x\in\Omega}$ be a family of functions with uniform (p,q)-structure, and let the function g be as in Definition 2.7. Suppose that $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ has the following properties*:

(i) *For each $\varepsilon > 0$, there is a $\sigma_\varepsilon \in L^{g,\kappa}_{\mathrm{loc}}(\Omega \cup \Gamma)$ such that*

$$\bigl|f(x,F) - g_x\bigl(|F|\bigr)\bigr| < \varepsilon g_x\bigl(|F|\bigr)$$

for every $x \in \Omega$ and $F \in \mathbb{R}^{N\times n}$ satisfying $|F| > \sigma_\varepsilon(x)$.

(ii) *There is an $L \geqslant 1$ and a function $\alpha \in L^{1,\kappa}_{\mathrm{loc}}(\Omega \cup \Gamma)$ such that*

$$\bigl|f(x,F)\bigr| \leqslant Lg\bigl(|F|\bigr) + \alpha(x)$$

for all $x \in \Omega$ and $F \in \mathbb{R}^{N\times n}$.

(iii) *For every $x, y \in \Omega$ and $\varepsilon > 0$, the inequality*

$$\bigl|f(x,F) - f(y,F)\bigr| \leqslant \bigl(\delta\bigl(|x-y|\bigr) + \varepsilon\bigr)g\bigl(|F|\bigr) + \bigl|\zeta_\varepsilon(x)\bigr| + \bigl|\zeta_\varepsilon(y)\bigr|$$

holds whenever $|F| > \min\{\sigma_\varepsilon(x), \sigma_\varepsilon(y)\}$.

Suppose that $u \in W^{1,1}(\Omega;\mathbb{R}^N)$ is a $(K,\{\omega_\varepsilon\},\{\nu_\varepsilon\})$-minimizer for the functional $K : W^{1,1}(\Omega;\mathbb{R}^N) \to \mathbb{R}^*$ defined by

$$K(w) := \int_\Omega f(x,\nabla w)\,\mathrm{d}x,$$

and that $u = \bar{u}$ on Γ in the sense of trace. Then $u \in W^{1,(p,\kappa)}_{\mathrm{loc}}(\Omega \cup \Gamma;\mathbb{R}^N)$ and $\nabla u \in L^{g,\kappa}_{\mathrm{loc}}(\Omega \cup \Gamma;\mathbb{R}^{N\times n})$.

Remark 6.1. For any $R < \infty$, one can impose the condition $\sigma_\varepsilon \geqslant R$ without affecting the required regularity for σ_ε. Hence we see that we need not require any continuity for $f(\cdot,F)$ when $|F| \leqslant R$.

Proof of Theorem 6.1. Let $v := u - \bar{u}$. It suffices to show that v is an almost minimizer for a family of functionals satisfying the hypotheses of Theorem 5.2. For each $y \in \Omega$, define the function $f_y : \mathbb{R}^{N\times n} \to \mathbb{R}$ by $f_y(F) := f(y,F)$, and define the functional $K_y : W^{1,1}(\Omega;\mathbb{R}^N) \to \mathbb{R}^*$ by

$$K_y(w) := \int_\Omega f_y(\nabla w + \nabla\bar{u})\,\mathrm{d}x.$$

By (ii), $\{f_y\}_{y\in\Omega}$ satisfies the growth condition required to use Theorem 5.2, and by (i), we have that $\{f_y\}_{y\in\Omega}$ is $L^{g,\kappa}_{\mathrm{loc}}(\Omega \cup \Gamma)$-asymptotically related to the family of functions $\{g_y\}_{y\in\Omega}$, which has uniform (p,q)-structure.

We will now show that v is a $(K_y,\{\gamma_\varepsilon\},\{\beta_\varepsilon\})$-minimizer for appropriate choices of $\{\gamma_\varepsilon\}_{\varepsilon>0}$ and $\{\beta_\varepsilon\}_{\varepsilon>0}$. Let $\varepsilon > 0$ and $\varphi \in W^{1,1}_0(\Omega \cap \mathcal{B}_{y,\rho};\mathbb{R}^N)$ be given. Since u is a $(K,\{\omega_\varepsilon\},\{\nu_\varepsilon\})$-minimizer, we have

$$K_y(v) \leqslant K_y(v+\varphi) + K(u+\varphi) - K_y(v+\varphi) - K(u) + K_y(v) + \int_{\Omega\cap\mathcal{B}_{y,\rho}} \bigl\{\bigl|\nu_\varepsilon(x)\bigr| + \bigl|\nu_\varepsilon(y)\bigr|\bigr\}\,\mathrm{d}x$$

$$+ \bigl(\omega_\varepsilon(\rho) + \varepsilon\bigr) \int_{\Omega\cap\mathcal{B}_{y,\rho}} \bigl\{\bigl|f(x,\nabla u)\bigr| + \bigl|f(x,\nabla u + \nabla\varphi)\bigr|\bigr\}\,\mathrm{d}x.$$

Recalling the definitions of K and K_y and the fact that $u = v + \bar{u}$, we get that

$$K_y(v) \leqslant K_y(v+\varphi) + \int_{\Omega\cap\mathcal{B}_{y,\rho}} \left|f(x,\nabla u+\nabla\varphi) - f(y,\nabla u+\nabla\varphi)\right| \mathrm{d}x$$
$$+ \int_{\Omega\cap\mathcal{B}_{y,\rho}} \left|f(x,\nabla u) - f(y,\nabla u)\right| \mathrm{d}x + \int_{\Omega\cap\mathcal{B}_{y,\rho}} \left\{\left|\nu_\varepsilon(x)\right| + \left|\nu_\varepsilon(y)\right|\right\} \mathrm{d}x$$
$$+ \left(\omega_\varepsilon(\rho)+\varepsilon\right) \int_{\Omega\cap\mathcal{B}_{y,\rho}} \left\{\left|f(x,\nabla u)\right| + \left|f(x,\nabla u+\nabla\varphi)\right|\right\} \mathrm{d}x.$$

Using (ii) and (iii), we see that

$$\left|f(x,F) - f(y,F)\right| \leqslant Lg\left(\sigma_\varepsilon(x)\right) + \alpha(x) + Lg\left(\sigma_\varepsilon(y)\right) + \alpha(y)$$
$$+ \left(\delta\left(|x-y|\right)+\varepsilon\right)g\left(|F|\right) + \left|\zeta(x)\right| + \left|\zeta(y)\right|$$

for all $x, y \in \Omega$ and $F \in \mathbb{R}^{N\times n}$. Also, using (i), (ii), and Definition 2.7 we obtain a constant c such that $|f(x,F)| \leqslant Lg(\sigma_1) + \alpha(x) + 2cg_y(|F|)$ for all $(x,F) \in \Omega \times \mathbb{R}^{N\times n}$. Using these estimates yields

$$K_y(v) \leqslant K_y(v+\varphi) + c_1 \int_{\Omega\cap\mathcal{B}_{y,\rho}} \left\{\hat{\beta}_\varepsilon(x) + \hat{\beta}_\varepsilon(y)\right\} \mathrm{d}x$$
$$+ c_1\left(\delta(\rho)+\omega_\varepsilon(\rho)+\varepsilon\right) \int_{\Omega\cap\mathcal{B}_{y,\rho}} \left\{g_y\left(|\nabla u|\right) + g_y\left(|\nabla u+\nabla\varphi|\right)\right\} \mathrm{d}x,$$

where we have defined $\hat{\beta}_\varepsilon := g(\sigma_\varepsilon) + \zeta_\varepsilon + b + (\omega_\varepsilon(\operatorname{diam}\Omega)+\varepsilon)g(\sigma_1)$, and where c_1 depends on L and c. Now using Lemma 3.2 and part (iv) of 3.1, we obtain a constant c_2 such that

$$K_y(v) \leqslant K_y(v+\varphi) + (\gamma_{c_2\varepsilon} + c_2\varepsilon) \int_{\Omega\cap\mathcal{B}_{y,\rho}} \left\{f_y\left(|\nabla u|\right) + f_y\left(|\nabla u+\nabla\varphi|\right)\right\} \mathrm{d}x$$
$$+ \int_{\Omega\cap\mathcal{B}_{y,\rho}} \left\{\beta_{c_2\varepsilon}(x) + \beta_{c_2\varepsilon}(y)\right\} \mathrm{d}x,$$

where we have put $\gamma_\varepsilon := c_2(\delta + \omega_{\varepsilon/c_1'})$ and $\beta_\varepsilon := c_1\hat{\beta}_\varepsilon + (2L+1)g(\sigma_{1/2}) + 2\alpha$. Note that $\beta_\varepsilon \in L^{1,\kappa}(\Omega)$ and that $\gamma_\varepsilon \in \mathcal{C}^0([0,\infty))$ is nondecreasing and satisfies $\gamma_\varepsilon(0) = 0$. Therefore we can use Theorem 5.2 to deduce that $u \in W^{1,(p,\kappa)}_{\mathrm{loc}}(\Omega\cup\Gamma;\mathbb{R}^N)$ and $\nabla u \in L^{g,\kappa}_{\mathrm{loc}}(\Omega\cup\Gamma;\mathbb{R}^N)$. $\square$

7. An application to partial differential equations

We now provide an application of our results that establishes Sobolev–Morrey regularity for weak solutions of partial differential equations. The strategy for the proof is to show that the solution u is a $(J,\{\omega_\varepsilon\},\{\nu_\varepsilon\})$-minimizer for some appropriate J, $\{\omega_\varepsilon\}_{\varepsilon>0}$, and $\{\nu_\varepsilon\}_{\varepsilon>0}$ so that we can apply Theorem 6.1.

Theorem 7.1. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with $\mathcal{C}^1$ boundary. Suppose that $\bar{u} \in W^{1,p}(\Omega;\mathbb{R}^N)$ satisfies $\nabla\bar{u} \in L^{g,\kappa}(\Omega;\mathbb{R}^{N\times n})$. Let $\{g_x\}_{x\in\Omega}$ be a family of functions with uniform (p,q)-structure, and let g be as in Definition 2.7. Suppose that $\mathcal{A}: \Omega\times\mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ and $h: \Omega\times\mathbb{R}^{N\times n} \to \mathbb{R}^N$ satisfy the following properties.*

(i) *For each $\varepsilon > 0$, there is a $\sigma_\varepsilon \in L^{\frac{p(q-1)}{p-1},\kappa}(\Omega)$ with $0 \leqslant \kappa < n$ such that for every $x \in \Omega$,*

$$\left|\mathcal{A}(x,F) - \frac{\partial}{\partial F} g_x\big(|F|\big)\right| < \varepsilon g_x'\big(|F|\big)$$

whenever $F \in \mathbb{R}^{N\times n}$ satisfies $|F| > \sigma_\varepsilon(x)$.

(ii) *There is a constant $L \geqslant 1$ and a function $\alpha \in L^{\frac{p}{p-1},\kappa}(\Omega)$ such that*

$$\big|\mathcal{A}(x,F)\big| \leqslant L|F|^{q-1} + \alpha(x) \quad \text{and}$$

$$\big|h(x,F)\big| \leqslant L\big[g_x\big(|F|\big)\big]^{\frac{p-1}{p}} + \alpha(x)$$

for all $x \in \Omega$ and $F \in \mathbb{R}^{N\times n}$.

(iii) *There are families $\{\zeta_\varepsilon\}_{\varepsilon>0} \subset L^{1,\kappa}(\Omega)$ and $\{\tau_\varepsilon\}_{\varepsilon>0} \subset L^{g,\kappa}(\Omega)$, along with a nondecreasing $\delta \in \mathcal{C}^0([0,\infty))$ satisfying $\delta(0) = 0$, such that*

$$\big|g_x\big(|F|\big) - g_y\big(|F|\big)\big| \leqslant \big(\delta\big(|x-y|\big) + \varepsilon\big) g\big(|F|\big) + \zeta_\varepsilon(x) + \zeta_\varepsilon(y)$$

for all $x, y \in \Omega$ and $F \in \mathbb{R}^{N\times n}$ such that $|F| \geqslant \min\{\tau_\varepsilon(x), \tau_\varepsilon(y)\}$.

Suppose that $u \in \bar{u} + W_0^{1,p}(\Omega;\mathbb{R}^N)$ is such that $g(|\nabla u|) \in L^1(\Omega)$, and also satisfies

$$\int_\Omega \mathcal{A}(x,\nabla u) : \nabla\varphi \, \mathrm{d}x = \int_\Omega h(x,\nabla u)\cdot\varphi \, \mathrm{d}x \tag{66}$$

whenever $\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^N)$ and $g(|\nabla\varphi|) \in L^1(\Omega)$. Then $u \in W^{1,(p,\kappa)}(\Omega;\mathbb{R}^N)$ and $\nabla u \in L^{g,\kappa}(\Omega;\mathbb{R}^{N\times n})$.

Proof. Define $f : \Omega \times \mathbb{R}^{N\times n}$ by $f(x,F) := g_x(|F|)$. Let the functional $J : W^{1,1}(\Omega;\mathbb{R}^N) \to \mathbb{R}^*$ be defined by $J(w) = \int_\Omega f(x,\nabla u)\,\mathrm{d}x$. In order to use Theorem 6.1, the only nontrivial thing to show is that u is an almost minimizer for J. With this end in mind, fix $x_0 \in \Omega$, $0 < \varepsilon < 1$ and $0 < \rho < \operatorname{diam}(\Omega)$, and let $\varphi \in W_0^{1,1}(\Omega \cap \mathcal{B}_{x_0,\rho}, \mathbb{R}^N)$ be given. If $g(|\nabla\varphi|) \notin L^1(\Omega)$, then using Lemma 3.2, we see that the inequality in (4) is trivially satisfied. So we may assume that $g(|\nabla\varphi|) \in L^1(\Omega)$. For convenience, define $v := u + \varphi$. Then using the definitions of J and f and the fact that $\nabla u = \nabla v$ outside $\Omega \cap \mathcal{B}_{x_0,\rho}$, we have that

$$\begin{aligned} J(u) - J(v) = &\int_{\Omega\cap\mathcal{B}_{x_0,\rho}} \left\{ g_x\big(|\nabla u|\big) - g_x\big(|\nabla v|\big) + \frac{\partial}{\partial F} g_x\big(|\nabla u|\big) : [\nabla v - \nabla u] \right\} \mathrm{d}x \\ &- \int_{\Omega\cap\mathcal{B}_{x_0,\rho}} \frac{\partial}{\partial F} g_x\big(|\nabla u|\big) : [\nabla v - \nabla u]\,\mathrm{d}x. \end{aligned}$$

By the convexity of g_x for each x, the first integral on the right is less than or equal to zero, and therefore, using (66), we have that

$$\begin{aligned} J(u) - J(v) &\leqslant - \int_{\Omega\cap\mathcal{B}_{x_0,\rho}} \frac{\partial}{\partial F} g_x\big(|\nabla u|\big) : [\nabla v - \nabla u]\,\mathrm{d}x \\ &= - \int_{\Omega\cap\mathcal{B}_{x_0,\rho}} \left\{ \frac{\partial}{\partial F} g_x\big(|\nabla u|\big) - \mathcal{A}(x,\nabla u) \right\} : [\nabla v - \nabla u]\,\mathrm{d}x - \int_{\Omega\cap\mathcal{B}_{x_0,\rho}} h(x,\nabla u)\cdot(v-u)\,\mathrm{d}x \\ &= I_1 + I_2, \end{aligned}$$

where I_1 and I_2 are defined to be the first and second integrals, respectively. We will estimate I_1 first, which we do by splitting $\Omega \cap \mathcal{B}_{x_0,\rho}$ into the set on which $|\nabla u| \leqslant \sigma_\varepsilon$ (call this set S and the corresponding integral $I_{1,S}$), and the set on which $|\nabla u| > \sigma_\varepsilon$ (call this set T and the corresponding integral $I_{1,T}$). Using Young's inequality and the growth conditions on $\mathcal{A}$ given in (ii), we obtain

$$I_{1,S} \leqslant c_\varepsilon \int\limits_S \big(g_x'(\sigma_\varepsilon) + L\sigma_\varepsilon^{q-1} + \alpha\big)^{\frac{p}{p-1}}\,\mathrm{d}x + \varepsilon \int\limits_S |\nabla\varphi|^p\,\mathrm{d}x.$$

Without loss of generality, we may assume $\sigma_\varepsilon(x) \geqslant 1$ for all x. By Lemma 3.1(i), we have that $g_x'(\sigma_\varepsilon) \leqslant q g_x(\sigma_\varepsilon)/\sigma_\varepsilon$, and so by Lemma 3.1(iii) and Definition 2.7, we see that $g_x'(\sigma_\varepsilon) \leqslant qcg(1)\sigma_\varepsilon^{q-1}$. Inserting this inequality into the previous estimate and using Lemma 3.1(iv) on the second integrand, we see that

$$I_{1,S} \leqslant c_{1,\varepsilon} \int\limits_S \big\{\sigma_\varepsilon^{\frac{p(q-1)}{p-1}} + \alpha^{\frac{p}{p-1}}\big\}\,\mathrm{d}x + c_2\varepsilon \int\limits_S g_x\big(|\nabla\varphi|\big)\,\mathrm{d}x, \tag{67}$$

where $c_{1,\varepsilon}$ depends on p, q, c, L, and $g(1)$, and c_2 depends on c and $g(1)$.

To estimate $I_{1,T}$, we employ assumption (i) and Lemma 3.1(vi), which gives

$$I_{1,T} \leqslant \varepsilon \int\limits_T g_x'\big(|\nabla u|\big)|\nabla\varphi|\,\mathrm{d}x \leqslant C\varepsilon \int\limits_T \big\{g_x\big(|\nabla u|\big) + g_x\big(|\nabla\varphi|\big)\big\}\,\mathrm{d}x. \tag{68}$$

Using Lemma 3.1(v) and combining (67) and (68) gives

$$I_1 \leqslant c_{1,\varepsilon} \int\limits_{\Omega\cap\mathcal{B}_{x_0,\rho}} \big\{\sigma_\varepsilon^{\frac{p(q-1)}{p-1}} + \alpha^{\frac{p}{p-1}}\big\}\,\mathrm{d}x + c_3\varepsilon \int\limits_{\Omega\cap\mathcal{B}_{x_0,\rho}} \big\{g_x\big(|\nabla u|\big) + g_x\big(|\nabla v|\big)\big\}\,\mathrm{d}x.$$

Now we will estimate I_2. Using Young's and Poincaré's inequalities, we obtain

$$I_2 \leqslant \varepsilon \int\limits_{\Omega\cap\mathcal{B}_{x_0,\rho}} \big|h(x,\nabla u)\big|^{\frac{p}{p-1}}\,\mathrm{d}x + c_\varepsilon\rho^p \int\limits_{\Omega\cap\mathcal{B}_{x_0,\rho}} |\nabla\varphi|^p\,\mathrm{d}x.$$

Using the growth condition on h given in (iii) and Lemma 3.1(iv), we have

$$I_2 \leqslant C\varepsilon \int\limits_{\Omega\cap\mathcal{B}_{x_0,\rho}} \big\{L^{\frac{p}{p-1}} g_x\big(|\nabla u|\big) + \alpha^{\frac{p}{p-1}}\big\}\,\mathrm{d}x + c_\varepsilon'\rho^p \int\limits_{\Omega\cap\mathcal{B}_{x_0,\rho}} \big\{g_x\big(|\nabla u|\big) + 1\big\}\,\mathrm{d}x.$$

Therefore, upon collecting our estimates for I_1 and I_2, we have that

$$I_1 + I_2 \leqslant c_{1,\varepsilon}' \int\limits_{\Omega\cap\mathcal{B}_{x_0,\rho}} \big\{\sigma_\varepsilon^{\frac{p(q-1)}{p-1}} + \alpha^{\frac{p}{p-1}}\big\}\,\mathrm{d}x + \big(c_\varepsilon'\rho^p + c_3\varepsilon\big) \int\limits_{\Omega\cap\mathcal{B}_{x_0,\rho}} \big\{g_x\big(|\nabla u|\big) + g_x\big(|\nabla v|\big)\big\}\,\mathrm{d}x.$$

Taking $\omega_\varepsilon(\rho) := c_{\varepsilon/c_3}'\rho^p$ and $\nu_\varepsilon := c_{1,\varepsilon/c_3}'(\sigma_{\varepsilon/c_3}^{\frac{p(q-1)}{p-1}} + \alpha^{\frac{p}{p-1}})$, we see that $\nu_\varepsilon \in L^{1,\kappa}(\Omega)$, $\omega_\varepsilon \in \mathcal{C}^0([0,\infty))$, and $\omega_\varepsilon(0) = 0$. Recalling the definition of f, we conclude that

$$J(u) - J(v) \leqslant \big(\omega_{c_3\varepsilon}(\rho) + c_3\varepsilon\big) \int\limits_{\Omega\cap\mathcal{B}_{x_0,\rho}} \big\{\big|f(x,\nabla u)\big| + \big|f(x,\nabla v)\big|\big\}\,\mathrm{d}x + \int\limits_{\Omega\cap\mathcal{B}_{x_0,\rho}} \big|\nu_{c_3\varepsilon}(x)\big|\,\mathrm{d}x,$$

and hence u is a $(J, \{\omega_\varepsilon\}, \{\nu_\varepsilon\})$-minimizer at x_0. Since $x_0 \in \Omega$ was arbitrary, we deduce from Theorem 6.1 that $u \in W^{1,(p,\kappa)}(\Omega; \mathbb{R}^N)$ and $\nabla u \in L^{g,\kappa}(\Omega; \mathbb{R}^N)$. $\square$

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